

Dynamic Managerial Compensation: A Variational Approach*

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Abstract

We study the optimal dynamics of incentives for a manager whose ability to generate cash flows changes stochastically with time and is his private information. We show that distortions (aka, wedges) under optimal contracts may either increase or decrease over time. In particular, when the manager's risk aversion and ability persistence are small, distortions decrease, on average, over time. For sufficiently high degrees of risk aversion and ability persistence, instead, distortions increase, on average, with tenure. Our results follow from a novel variational approach that permits us to tackle directly the "full program," thus bypassing some of the difficulties of the "first-order approach" encountered in the dynamic mechanism design literature.

JEL classification: D82

Keywords: managerial compensation, incentives, pay for performance, dynamic mechanism design, adverse selection, moral hazard, persistent productivity shocks, risk aversion, wedges, variational approach, first-order approach.

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1 Introduction

In dynamic business environments, the ability of top managers to generate profits for their firms is expected to change with time as a result, for example, of changes in the organization, the arrival of new technologies, or market consolidations. A key difficulty is that, while such changes are largely expected, their implications for profitability typically remain the managers' private information. In this paper we ask the following questions: are managers induced to work harder at the beginning of their employment relationships or later on? Do the distortions in the provision of incentives due to asymmetric information tend to decrease over time? How does "pay for performance" change over the course of the employment relationship to sustain the desired dynamics of effort? Should the intertemporal variation in the provision of incentives be more pronounced for managers of low or of high initial productivity?

We consider an environment where, at the time of joining the firm, the managers possess private information about their productivity (i.e., their ability to generate cash flows). This private information originates, for instance, in tasks performed in previous contractual relationships, as well as in personal traits that are not directly observable by the firm. The purpose of the analysis is to examine the implications of this private information, and the fact that it evolves with time, for the dynamic provision of incentives.

In the environment described above, a firm finds it expensive to ask a manager to exert more effort for three reasons. First, higher effort is costly for the manager and must be compensated. Second, asking higher effort of a manager with a given productivity requires increasing the compensation promised to all managers with higher productivity. This compensation is required even if the firm does not ask the more productive managers to exert more effort and represents an additional "rent" for these managers. It is needed to discourage them from mimicking the less productive managers by misrepresenting their productivity and reducing their effort. Third, inducing higher effort requires pay to be more sensitive to performance. This, in turn, exposes the managers to more volatility in their compensation. When the managers are risk averse, this increase in volatility reduces their expected payoff, requiring higher compensation by the firm.

The above effects of effort on compensation shape the way the firm induces its managers to respond to productivity shocks over time. In this paper we investigate the implications of the above trade-offs both for the dynamics of effort and for the distortions in the provision of incentives due to asymmetric information. As in the new Dynamic Public Finance, in the presence of wealth effects (that is, beyond the quasilinear case), distortions are best measured by the "wedge" between the marginal cash flows generated by higher effort and the marginal compensation that must be paid to the managers to keep their utility constant. Importantly, if one considers compensation schemes that are differentiable in the firm's cash flows and depend only on (a) the history of reported productivity and (b) the cash flows generated in the period of compensation, then the wedges are also related

to the “local” sensitivity of compensation to cash flows around the “equilibrium cash flows”. More generally, the dynamics of wedges provides information on how the firm optimally distorts both effort and compensation intertemporally to reduce the managers’ information rents.

Our analysis identifies certain properties of optimal contracts by applying variational arguments directly to the firm’s “full problem”. That is, we directly account for all of the manager’s incentive constraints. For any incentive-compatible contract, we identify certain “admissible perturbations” that *preserve* participation and incentive-compatibility constraints. For a contract to be optimal, these perturbations must not increase the firm’s expected profits. This requirement implies a new set of Euler conditions that equate the average marginal benefit of higher effort with its average marginal cost. The average marginal benefit is simply the increase in the firm’s expected cash flows. The average marginal cost combines the disutility of effort with (a) the cost of increasing the compensation for higher types to induce them to reveal their private information, and (b) the cost of increasing the volatility of compensation in case the manager is risk averse. Importantly, the admissible variations that lead to the Euler conditions do not permit us to characterize how effort and compensation respond to all possible contingencies. However, they do permit us to identify certain predictions as to how, on average, effort and the power of incentives evolve over time under fully optimal contracts.

The advantage of this approach is that it permits us to bypass some of the difficulties encountered in the literature. The typical approach involves imposing only a restricted set of incentive constraints, usually referred to as “local” constraints. In other words, one first solves a “relaxed problem”. One then seeks to identify restrictions on the primitive environment that guarantee that the solution to the relaxed problem satisfies the remaining incentive constraints.¹ When validated, the relaxed approach has the advantage of yielding *ex-post predictions* about effort and compensation that depend on the realized productivity history. In contrast, the variational approach we develop here yields only *ex-ante predictions* that hold by averaging over productivity histories.² The primitive conditions under which the variational approach yields useful predictions are neither stronger nor weaker than the conditions that validate the first-order approach. For example, while the variational approach requires (a) the disutility of effort to be quadratic and (b) effort to possibly take negative values (so as to avoid corner solutions), such restrictions are not required under the first-order approach. On the other hand, some of the restrictions on the productivity distribution required by the first-order approach can be dispensed with under the variational approach.

¹The relaxed approach fails whenever the effort policies that solve the relaxed problem fail to satisfy certain “monotonicity conditions” necessary for incentive compatibility (for the present paper, see Condition (B) in Proposition 1). We refer the reader to Pavan, Segal, and Toikka (2014) for further discussion of how the relaxed approach may fail in quasilinear settings.

²Predictions that hold only on average may still be important for empirical work, especially given that histories of productivity shocks are typically unobservable to the econometrician.

Key results. Consider first the case where managers are risk neutral. The concern for reducing the rent left to those managers whose initial productivity is high typically leads the firm to distort downward (relative to the first best) the level of effort asked of those managers whose initial productivity is low. While a similar property has been noticed in previous work (see, among others, Laffont and Tirole, 1986), all existing results have been established for cases where the optimal contract is the solution to the “relaxed program”. We show that this property is true more generally, as long as the effort that the firm asks at each point in time is bounded away from zero from below with probability one (that is, except over at most a zero-measure set of productivity histories). We also provide novel primitive conditions for this to be the case.

An important further result is that, whenever (a) on average, period-1 effort is distorted downward relative to the first-best level, and (b) the effect of the initial productivity on future productivity declines with time, the firm asks, on average, for higher effort later in the relationship. This is because, when productivity is less than fully persistent, the benefit of distorting the effort of those managers whose initial productivity is low so as to reduce the compensation paid to those managers whose initial productivity is high is greatest early in the relationship.

Next consider the case where the managers are risk averse. Mitigating the volatility of future compensation calls for contracts that, on average, further distort effort and compensation away from their efficient levels later in the relationship. The reason is that, viewed from the date the contract is initially agreed, managers face greater uncertainty about their productivity at later dates. Whether distortions increase or decrease, on average, over time then depends on the degrees of managerial risk-aversion and productivity persistence. For low degrees of risk aversion and low degrees of productivity persistence, the dynamics of distortions are the same as in the risk neutral case (that is, distortions decrease, on average, over time). When, instead, productivity is perfectly persistent (meaning that shocks to productivity are permanent as in the case of a random walk), then, *for any degree of risk aversion*, distortions increase, on average, over time.³ Subject to certain qualifications, we argue that the same result should also be expected for large degrees of persistence. In particular, we argue that the dynamics of distortions are continuous with respect to the degree of productivity persistence, provided that effort under optimal policies remains bounded.

Implications for empirical work. The empirical literature typically focuses on a measure of incentives proposed by Jensen and Murphy (1990). This is the responsiveness of CEO pay to changes in shareholder wealth. The empirical evidence of how incentives vary with tenure is mixed. Gibbons and Murphy (1992), Lippert and Porter (1997), and Cremers and Palia (2010) find that the sensitivity of managerial pay to performance typically increases with tenure, while Murphy (1986)

³Note that a process that is fully persistent is not necessarily one in which productivity is constant over time. The result that distortions increase, on average, over time in the random walk case, for any strictly concave felicity function, hinges on the fact that future productivity is stochastic.

and Hill and Phan (1991) find evidence of the opposite. A number of theories have been proposed to explain these patterns. Gibbons and Murphy (1992) provide a model of career concerns to suggest that explicit pay-for-performance ought to increase closer to a manager’s retirement. Edmans et al. (2012) suggest a similar conclusion but based on the idea that, with fewer remaining periods ahead, replacing current pay with future promised utility becomes more difficult to sustain. Arguments for the opposite finding have often centered on the possibility that managers capture the board once their tenure has grown large (see, e.g., Hill and Phan (1991) and Bebchuk and Fried (2004)), while Murphy (1986) proposes a theory based on market learning about managerial quality over time, where the learning is symmetric between the market and the managers.

Our paper contributes to this debate by indicating that a key determinant for whether incentives (proxied by the sensitivity of pay to performance) ought to increase or decrease with tenure may be the manager’s *degree of risk aversion*. Another prediction of our model, although one which is subject to the limitations of the relaxed approach discussed above, is that, under risk neutrality, the increase in the provision of incentives over time is most pronounced for those managers whose initial productivity is low.⁴ Because productivity is positively correlated with performance, this result suggests a negative correlation between early performance and the increase in the provision of incentives (equivalently, in the sensitivity of pay for performance) over the course of the employment relationship. This prediction seems a distinctive feature of our theory, albeit one that, to the best of our knowledge, has not been tested yet.

Organization of the paper. The rest of the paper is organized as follows. We briefly review some pertinent literature in the next section. Section 3 describes the model while Section 4 characterizes the firm’s optimal contract. Section 5 concludes. All proofs are in the Appendix at the end of the manuscript.

2 Related literature

The literature on managerial compensation is too vast to be discussed within the context of this paper. We refer the reader to Prendergast (1999) for an excellent overview and to Edmans and Gabaix (2009) for a survey of some recent developments. Below, we limit our discussion to the papers that are most closely related to our own work.

Our work is related to the literature on “dynamic moral hazard” and its application to managerial compensation. Seminal works in this literature include Lambert (1983), Rogerson (1985), and Spear and Srivastava (1987). These works provide qualitative insights about optimal contracts but do not provide a full characterization. This has been possible only in restricted settings: Phelan and

⁴We expect that this property carries over to settings with risk-averse managers, provided that productivity is less than fully persistent (see Figure 2, for instance, for an example).

Townsend (1991) characterize optimal contracts numerically in a discrete-time model, while Sannikov (2008) characterizes the optimal contract in a continuous-time setting with Brownian shocks.⁵ In contrast to these works, Holmstrom and Milgrom (1987) show that the optimal contract has a simple structure when (a) the agent does not value the timing of payments, (b) noise follows a Brownian motion, and (c) the agent’s utility is exponential and defined over consumption net of the disutility of effort. Under these assumptions, the optimal contract takes the form of a simple linear aggregator of total profits.

Contrary to the above works, in the current paper we assume that, in each period, the manager observes the shock to his productivity before choosing effort.⁶ In this respect, our paper is closely related to Laffont and Tirole (1986) who first proposed this alternative timing. This timing permits one to use techniques from the mechanism design literature to solve for the optimal contract. The same approach has been recently applied to dynamic managerial compensation by Edmans and Gabaix (2011) and Edmans et al. (2012). Our model is similar in spirit, but with a few key distinctions. First, we assume that the manager is privately informed about his initial productivity before signing the contract; this is what drives the result that the manager must be given a strictly positive share of the surplus. A second key difference is that we characterize how effort and the power of incentives in the optimal contract evolve over time.⁷

Our paper is also related to our previous work on managerial turnover in a changing world (Garrett and Pavan, 2012). In that paper, we assume that all managers are risk neutral and focus on the dynamics of retention decisions. In contrast, in the present paper, we abstract from retention (i.e., assume a single manager) and focus instead on the effect of risk aversion on the dynamics of incentives.

A growing number of papers study optimal financial instruments in dynamic principal-agent relationships. For instance, DeMarzo and Sannikov (2006), DeMarzo and Fishman (2007), Sannikov (2007),⁸ and Biais et al. (2010) study optimal financial contracts for a manager who privately observes the dynamics of cash flows and can divert funds from investors to private consumption. In these papers, it is typically optimal to induce the highest possible effort (which is equivalent to no stealing/no saving); the instrument which is then used to create incentives is the probability of

⁵See also Sadzik and Stacchetti (2013) for recent work on the relationship between discrete-time and continuous-time models.

⁶We abstract from the possibility that performance is affected by transitory noise that occurs after the manager chooses his effort. It is often the case, however, that compensation can be structured so that it continues to implement the desired effort policies even when performance is affected by transitory noise.

⁷In contrast, the above work assumes that it is optimal to induce the highest feasible effort constantly over time, thus bypassing the difficulty of balancing the costs and benefits of additional effort in response to productivity shocks.

⁸As in our work, and contrary to the other papers cited here, Sannikov (2007) allows the agent to possess private information prior to signing the contract. Assuming the agent’s initial type can be either “bad” or “good”, he characterizes the optimal separating menu where only good types are funded.

terminating the project. One of the key findings is that the optimal contract can often be implemented using long-term debt, a credit line, and equity. The equity component represents a linear component to the compensation scheme which is used to make the agent indifferent as to whether or not to divert funds to private use. Since the agent's cost of diverting funds is constant over time and output realizations, so is the equity share. In contrast, we provide an explanation for why and how this share may change over time. While these papers suppose that cash-flows are i.i.d., Tchisty (2006) explores the consequences of correlation and shows that the optimal contract can be implemented using a credit line with an interest rate that increases with the balance. As in Tchisty (2006), we also assume that managerial productivity is imperfectly correlated over time.

From a methodological standpoint, we draw from recent results in the dynamic mechanism design literature. In particular, the necessary and sufficient conditions for incentive compatibility in Proposition 1 in the present paper adapt to the environment under examination results in Theorems 1 and 3 in Pavan, Segal, and Toikka (2014). That paper provides a general treatment of incentive compatibility in dynamic settings. It extends previous work by Baron and Besanko (1984), Besanko (1985), Courty and Li (2000), Battaglini (2005), Eso and Szentes (2007), and Kapicka (2013), among others, by allowing for more general payoffs and stochastic processes and by identifying the role of impulse responses as the key driving force for the dynamics of optimal contracts. One of the key properties identified in this literature is that of declining distortions (see, e.g., Baron and Besanko, 1984, Besanko, 1985, and Battaglini, 2005, among others). A contribution of the present paper is to qualify the extent to which this property is robust to the possibility that the agent is risk averse.⁹ In this respect, the paper is also related to Farinha Luz (2014) who, in an insurance model with two types, identifies conditions on the utility function that guarantee that distortions decrease over time over all possible paths. Another contribution of the present paper relative to this literature is in the way we identify certain properties of optimal contracts. As explained above, this involves identifying perturbations of the proposed policies that preserve participation and incentive-compatibility constraints and then using variational arguments to verify the key properties. To the best of our knowledge, this approach is new to the dynamic mechanism design literature. Variational methods have been used in agency models with hidden actions only by Cvitanic, Wan and Zhang (2009), Capponi, Cvitanic, and Yolcu (2012), and Sannikov (2014). For a general treatment of variational methods in optimization problem, see e.g. Luenberger (1997).

The paper is also related to the literature on optimal dynamic taxation (also known as Mirrleesian taxation, or new public finance). Recent contributions to this literature include Battaglini and Coate (2008), Zhang (2009), Golosov, Troshkin, and Tsyvinski (2012) and Farhi and Werning (2013). Our definition of distortions in the provision of incentives coincides with the definition of labor "wedge" in this literature, which is considered the appropriate measure of distortions in the provision of incentives

⁹For static models with risk aversion, see Salanie (1990), and Laffont and Rochet (1998).

in the presence of private information and non-quasilinear payoffs. A complication encountered in this literature is that, because of risk aversion, policies solving the relaxed program can only be computed numerically; likewise, the incentive-compatibility of such policies can only be checked with numerical methods. The approach introduced in the present paper may perhaps prove useful for characterizing certain properties of optimal dynamic taxes, as well as optimal contracts for risk-averse agents in other settings.

3 The Model

3.1 The environment

Players, actions, and information. The firm’s shareholders (hereafter referred to as the principal) hire a manager to work on a project for two periods. In each period $t = 1, 2$, the manager receives some private information $\theta_t \in \Theta_t = [\underline{\theta}_t, \bar{\theta}_t]$ about his ability to generate cash flows for the firm (his type). After observing θ_t , he then chooses effort $e_t \in E = \mathbb{R}$. The latter, combined with the manager’s productivity θ_t , then leads to cash flows π_t according to the simple technology $\pi_t = \theta_t + e_t$.

Both $\theta \equiv (\theta_1, \theta_2) \in \Theta \equiv \Theta_1 \times \Theta_2$ and $e \equiv (e_1, e_2) \in \mathbb{R}^2$ are the manager’s private information. Instead, the cash flows $\pi \equiv (\pi_1, \pi_2)$ are verifiable, and hence can be used as a basis for the manager’s compensation.

Payoffs. For simplicity, we assume no discounting.¹⁰ The principal’s payoff is the sum of the firm’s cash flows in the two periods, net of the manager’s compensation, i.e.

$$U^P(\pi, c) = \pi_1 + \pi_2 - c_1 - c_2,$$

where c_t is the period- t compensation to the manager and where $c \equiv (c_1, c_2)$. The function U^P is also the principal’s Bernoulli utility function used to evaluate possible lotteries over (π, c) .

By choosing effort e_t in period t , the manager suffers a disutility $\psi(e_t)$. The manager’s Bernoulli utility function is then given by

$$U^A(c, e) = v(c_1) + v(c_2) - \psi(e_1) - \psi(e_2), \tag{1}$$

where $v : \mathbb{R} \rightarrow \mathbb{R}$ is a strictly increasing, weakly concave, surjective, Lipschitz continuous, and differentiable function.¹¹ The case where v is linear corresponds to the case where the manager is risk neutral, while the case where v is strictly concave corresponds to the case where he is risk averse.

¹⁰None of the results hinge on this assumption.

¹¹The reason for assuming that $v(\cdot)$ is surjective is twofold: (i) it guarantees the existence of punishments sufficient to discourage the agent from not delivering the anticipated cash flows; (ii) it also guarantees that, given any effort policy that satisfies the appropriate monotonicity conditions of Proposition 1 below, one can always construct a compensation scheme that delivers, on path, the utility that is required for the agent to report his productivity truthfully.

Note that the above payoff specification also implies that the manager has preferences for consumption smoothing. This assumption is common in the dynamic moral hazard (and taxation) literature (a few notable exceptions are Holmstrom and Milgrom (1987) and more recently Edmans and Gabaix (2011)).¹² We denote the inverse of the felicity function by w (i.e., $w \equiv v^{-1}$).

Productivity process. The manager's first-period productivity, θ_1 , is drawn from an absolutely continuous c.d.f. F_1 with density f_1 strictly positive over Θ_1 . His second-period productivity is drawn from an absolutely continuous c.d.f. $F_2(\cdot|\theta_1)$ with density $f_2(\cdot|\theta_1)$ strictly positive over a subset $\Theta_2(\theta_1) = [\underline{\theta}_2(\theta_1), \bar{\theta}_2(\theta_1)]$ of Θ_2 . We will assume θ_t follows an autoregressive process so that $\tilde{\theta}_2 = \gamma\tilde{\theta}_1 + \tilde{\varepsilon}$, with $\tilde{\varepsilon}$ drawn from a continuously differentiable c.d.f. G with finite support $[\underline{\varepsilon}, \bar{\varepsilon}]$.¹³ That productivity follows an autoregressive process implies that the impulse responses of period-2 types, θ_2 , to period-1 types, θ_1 , are constant and equal to γ . While monotonicity of the impulse responses is often used to validate the first-order approach (see Pavan, Segal, and Toikka, 2014, and Battaglini and Lamba (2014)), it does not play a role in the variational approach in our paper. The perturbations we discuss in Section 4 continue to preserve incentive compatibility under more general processes in which impulse responses are neither constant nor monotone. The only result that uses the assumption that impulse responses are constant is Proposition 3 below where we consider perturbations of effort over multiple periods that leave the manager's payoff unchanged. However, this result is superseded by Proposition 2 whenever expected effort can be shown to be non-negative.

We assume that $\gamma \geq 0$, so that higher period-1 productivity leads to higher period-2 productivity in the sense of first-order stochastic dominance. We will refer to $\gamma = 1$ as to the case of "full persistence" (meaning that, holding effort fixed, the effect of any shock to period-1 productivity on the firm's average cash flows is constant over time). We will be primarily interested in the case where $\gamma \in [0, 1]$.

Effort disutility. As mentioned in the Introduction, the variational approach in the present paper requires that $\psi(e) = e^2/2$ for all e . That the disutility of effort is quadratic permits us to identify a convenient family of perturbations to incentive-compatible contracts that preserve incentive compatibility. That effort can take negative values in turn permits us to disregard the possibility of corner solutions. It also guarantees that a manager misreporting his productivity can always adjust his effort to "hide the lie" by generating the same cash flows as the type being mimicked. This property also facilitates the analysis by turning the model de facto into a pure adverse selection one, as first noticed by Laffont and Tirole (1986).

¹²As is standard, this specification presumes that the manager's period- t consumption c_t coincides with the period- t compensation. In other words, it abstracts from the possibility of secret private saving. The specification also presumes time consistency. This means that, in both periods, the manager maximizes the expectation of U^A , where the expectation depends on all available information.

¹³Throughout, we use the superscript "~" to denote random variables.

Many of the formulas below will retain the notation $\psi'(e)$ and $\psi''(e)$ to distinguish the role of these functions from effort e and from the constant 1.

3.2 The principal's problem

The principal's problem consists in choosing a contract specifying for each period a recommended effort choice along with compensation that conditions on the observed cash flows. It is convenient to think of such a contract as a mechanism $\Omega \equiv \langle \xi, x \rangle$ comprising a recommended *effort policy* $\xi \equiv (\xi_1(\cdot), \xi_2(\cdot))$ and a *compensation scheme* $x \equiv (x_1(\cdot), x_2(\cdot))$.

The effort $\xi_1(\theta_1)$ that the firm recommends in period one is naturally restricted to depend on the manager's self-reported productivities $\theta = (\theta_1, \theta_2)$ only through θ_1 . This property reflects the assumption that the manager learns his period-2 productivity θ_2 only at the beginning of the second period, as explained in more detail below.¹⁴ The effort that the firm recommends in the second period, $\xi_2(\theta)$, depends on the manager's self-reported productivity in each of the two periods, but is independent of the first-period cash flow, π_1 . This property can be shown to be without loss of optimality for the principal, a consequence of the assumptions that (i) cash flows are deterministic functions of effort and productivity (which implies that, on path, π_1 is a deterministic function of θ_1), and (ii) the manager is not protected by limited liability (which implies that incentives for period-1 effort can be provided through the compensation scheme $x_1(\cdot)$ without the need to condition effort in the second period on off-path cash flows). The compensation $x_t(\theta, \pi)$ paid in each period naturally depends both on the reported productivities and the observed cash flows.¹⁵ Note that, by reporting his productivity, the manager effectively induces a change to his compensation scheme. This seems consistent with the practice of managers proposing changes to their compensation, which has become quite common (see, among others, Bebchuk and Fried, 2004, and Kuhnen and Zwiebel, 2008).¹⁶

Let $\pi_t(\theta) \equiv \theta_t + \xi_t(\theta)$ denote the period- t "equilibrium" cash flows (by "equilibrium", hereafter we mean under a truthful and obedient strategy for the manager). Note that the compensation scheme x is defined for all possible cash flows $\pi \in \mathbb{R}^2$, not only the equilibrium ones; i.e., each payment $x_t(\theta, \pi)$ is defined also for $\pi \neq \pi(\theta) \equiv (\pi_1(\theta_1), \pi_2(\theta))$. For any $\theta \in \Theta$, we then further define $c_t(\theta) = x_t(\theta, \pi(\theta))$ to be the equilibrium compensation to the manager in state θ and refer to $c \equiv (c_1(\cdot), c_2(\cdot))$ as the firm's *compensation policy*. While our focus is on characterizing the firm's optimal effort and compensation policies, the role of the out-of-equilibrium payments $x_t(\theta, \pi)$

¹⁴While we naturally restrict ξ_1 to depend on θ only through the period-1 productivity θ_1 , we often abuse notation by writing $\xi_1(\theta)$ whenever this eases the exposition without the risk of confusion.

¹⁵Again, we abuse notation by writing $x_1(\theta, \pi)$ when convenient, although x_1 is naturally restricted to depend only on (θ_1, π_1) .

¹⁶However, note that the allocations sustained under the optimal contract as determined below are typically sustainable also without the need for direct communication between the manager and the firm (this is true, in particular, when there is a one-to-one mapping from the manager's productivity to the equilibrium cash flows).

for $\pi \neq \pi(\theta)$ is to guarantee that the manager finds it optimal to follow a truthful and obedient strategy, as will be discussed in detail below.

Importantly, we assume that the firm offers the manager the contract after he is already informed about his initial productivity $\theta_1 \in \Theta_1$. After receiving the contract, the manager then chooses whether or not to accept it. If he rejects it, he obtains an outside continuation payoff which we assume to be equal to zero for all possible types. If, instead, he accepts it, he is then bound to stay in the relationships for the two periods.¹⁷ He is then asked to report his productivity $\hat{\theta}_1 \in \Theta_1$ and is recommended effort $\xi_1(\hat{\theta}_1)$. The manager then privately chooses effort e_1 , which combines with the manager's productivity θ_1 to give rise to the period-1 cash flows $\pi_1 = \theta_1 + e_1$. After observing the cash flows π_1 , the firm then pays the manager a compensation $x_1(\hat{\theta}_1, \pi_1)$.

The functioning of the contract in period two parallels the one in period one. At the beginning of the period, the manager learns his new productivity θ_2 . He then updates the principal by sending a new report $\hat{\theta}_2 \in \Theta_2$. The contract then recommends effort $\xi_2(\hat{\theta})$ which may depend on the entire history $\hat{\theta} \equiv (\hat{\theta}_1, \hat{\theta}_2)$ of reported productivities. The manager then privately chooses effort e_2 which, together with θ_2 , leads to the cash flows π_2 . After observing π_2 , the firm then pays the manager a compensation $x_2(\hat{\theta}, \pi)$ and the relationship is terminated.

As usual, we restrict attention to contracts that are accepted by all types and that induce the manager to report truthfully and follow the principal's recommendations in each period.¹⁸ We will refer to such contracts as *individually rational* and *incentive compatible*.

4 Profit-maximizing Contracts

4.1 Implementable policies

As anticipated above, the principal's problem consists in choosing effort and compensation policies $\langle \xi, c \rangle$ to maximize the firm's expected profits subject to the policies being *implementable*. By this we mean the following.

¹⁷We do not expect our results to hinge on the assumption that the manager is constrained to stay in the relationship throughout both periods. For example, when the manager's period-2 outside option is sufficiently small, the period-2 individual rationality constraints are slack. One reason why the outside option in period two may be small is that the manager may anticipate adverse treatment by the labor market in case he leaves the firm prematurely. Fee and Hadlock (2004), for instance, document evidence for a labor market penalty in case a senior executive leaves the firm early, although the size of this penalty depends on the circumstances surrounding departure.

¹⁸Note that the manager's second-period payoff does not depend directly on his first-period productivity. Hence, the environment is "Markov". This means that restricting attention to contracts that induce the manager to follow a truthful and obedient strategy in period two also after having departed from truthful and obedient behavior in period one is without loss of optimality.

Definition 1 *The effort and compensation policies $\langle \xi, c \rangle$ are implementable if there exists a compensation scheme x such that (i) the contract $\Omega = \langle \xi, x \rangle$ is incentive compatible and individually rational, and (ii) the manager's on-path compensation under the contract $\Omega = \langle \xi, x \rangle$ is given by c , i.e. $x_t(\theta, \pi(\theta)) = c_t(\theta)$ for all t , and all θ .*

Our first result provides a complete characterization of implementable policies. For any $(\theta; \xi)$, let

$$W(\theta; \xi) \equiv \psi(\xi_1(\theta_1)) + \psi(\xi_2(\theta)) + \int_{\theta_1}^{\theta_1} \left\{ \psi'(\xi_1(s)) + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[\psi'(\xi_2(s, \tilde{\theta}_2)) \right] \right\} ds \quad (2)$$

$$+ \int_{\theta_2}^{\theta_2} \psi'(\xi_2(\theta_1, s)) ds - \mathbb{E}^{\tilde{\theta}_2|\theta_1} \left[\int_{\theta_2}^{\tilde{\theta}_2} \psi'(\xi_2(\theta_1, s)) ds \right].$$

Proposition 1 *The effort and compensation policies $\langle \xi, c \rangle$ are implementable if and only if the following conditions jointly hold: (A) for all $\theta \in \Theta$,*

$$v(c_1(\theta_1)) + v(c_2(\theta)) = W(\theta; \xi) + K \quad (3)$$

where $K \geq 0$ is such that

$$\mathbb{E}^{\tilde{\theta}|\theta_1} \left[W(\tilde{\theta}; \xi) - \psi(\xi_1(\tilde{\theta}_1)) - \psi(\xi_2(\tilde{\theta})) \right] + K \geq 0 \quad (4)$$

for all θ_1 ; and (B)(i) for all $\theta_1, \hat{\theta}_1 \in \Theta_1$,

$$\int_{\hat{\theta}_1}^{\theta_1} \left\{ \psi'(\xi_1(\hat{\theta}_1) + \hat{\theta}_1 - s) + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[\psi'(\xi_2(\hat{\theta}_1, \tilde{\theta}_2)) \right] \right\} ds$$

$$\leq \int_{\hat{\theta}_1}^{\theta_1} \left\{ \psi'(\xi_1(s)) + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[\psi'(\xi_2(s, \tilde{\theta}_2)) \right] \right\} ds, \quad (5)$$

and B(ii) $\pi_1(\theta_1) + \gamma \mathbb{E}^{\tilde{\theta}_2|\theta_1} \left[\pi_2(\theta_1, \tilde{\theta}_2) \right]$ is non-decreasing in θ_1 and, for all $\theta_1 \in \Theta_1$, $\pi_2(\theta_1, \theta_2)$ is non-decreasing in θ_2 .

Note that Condition (A) says that the manager's ex-post equilibrium payoff

$$V(\theta) \equiv v(c_1(\theta_1)) + v(c_2(\theta)) - \psi(\xi_1(\theta_1)) - \psi(\xi_2(\theta))$$

in each state of the world $\theta = (\theta_1, \theta_2)$ must be equal to his period-1 expected payoff

$$\mathbb{E}^{\tilde{\theta}|\theta_1} [V(\tilde{\theta})] = \mathbb{E}^{\tilde{\theta}|\theta_1} [V(\tilde{\theta})] + \int_{\theta_1}^{\theta_1} \left\{ \psi'(\xi_1(s)) + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[\psi'(\xi_2(s, \tilde{\theta}_2)) \right] \right\} ds \quad (6)$$

augmented by a term

$$\int_{\theta_2}^{\theta_2} \psi'(\xi_2(\theta_1, s)) ds - \mathbb{E}^{\tilde{\theta}_2|\theta_1} \left[\int_{\theta_2}^{\tilde{\theta}_2} \psi'(\xi_2(\theta_1, s)) ds \right]$$

that guarantees that the manager has the incentives to report truthfully not only in period-1 but also in period-2 and that vanishes when computed based on period 1's private information, θ_1 . The necessity of this condition is obtained by combining certain period-2 local necessary conditions for incentive compatibility (as derived, for example, in Laffont and Tirole (1986)) with certain period-1 local necessary conditions for incentive compatibility (as derived, for example, in Pavan, Segal and Toikka (2014); see also Garrett and Pavan (2012) for a similar derivation in a model of managerial turnover, and see Garrett and Pavan (2011) for a generalization of this condition to a richer setting with more than two periods). Observe that Condition (A) in the proposition implies that the surplus that type θ_1 expects above the one expected by the lowest period-1 type $\underline{\theta}_1$ is increasing in the effort that the firm asks of managers with initial productivities $\theta'_1 \in (\underline{\theta}_1, \theta_1)$ in each of the two periods. This surplus is necessary to dissuade type θ_1 from mimicking the behavior of these lower types. Such mimicry would involve, say, reporting a lower type in the first period and then replicating the distribution of that type's productivity reports in the second period. By replicating the same cash flows expected from a lower type, a higher type obtains the same compensation while working less if the effort asked of the lower type is positive, and more if the effort asked of the lower type is negative.

Also note that, when productivity is only partially persistent (in the autoregressive model, when $\gamma < 1$), then asking for a lower period-1 effort from types $\theta'_1 < \theta_1$ is more effective in reducing type θ_1 's expected surplus than asking for a lower period-2 effort from the same types. The reason is that the amount of effort that type θ_1 expects to be able to save relative to these lower period-1 types (alternatively, the extra effort that he must provide, in case the effort asked to these lower types is negative) is smaller in the second period, reflecting the fact that the initial productivity is imperfectly persistent. As we will see below, this property plays an important role in shaping the dynamics of effort and the distortions in the provision of incentives under optimal contracts.

Finally note that the scalar K in (3) corresponds to the expected payoff $\mathbb{E}^{\tilde{\theta}|\underline{\theta}_1}[V(\tilde{\theta})]$ of the lowest period-1 type. Using (6), it is easy to see that, when the effort requested is always non-negative, then if the lowest period-1 type finds it optimal to accept the contract, then so does any manager whose initial productivity is higher. This property, however, need not hold in case the firm requests a negative effort from a positive-measure set of types.

Next consider Condition (B) in the proposition. Observe that, while Condition (A) imposes restrictions on the compensation that must be paid to the manager, for given effort policy ξ , Condition (B) imposes restrictions on the effort policy that are independent of the manager's felicity function, v . In particular, Condition (B)(ii) combines the familiar monotonicity constraint for the second-period cash flows from static mechanism design (e.g., Laffont and Tirole (1986)) with a novel monotonicity constraint that requires the NPV of the expected cash-flows, weighted by the impulse responses (which here are equal to one in the first period and γ in the second period) to be non-decreasing in

period-1 productivity.¹⁹ Finally, Condition (B)(i) is an “integral monotonicity condition,” analogous to the one in Theorem 3 of Pavan, Segal and Toikka (2014). That the conditions in the proposition are necessary follows from arguments similar to those in Theorems 2 and 3 in Pavan, Segal, and Toikka (2014), adapted to the environment under examination here. That they are also sufficient follows from the fact that, when satisfied, one can construct compensation schemes under which the best a manager can do when mimicking a different type is to replicate the same cash flows of the type being mimicked. This turns the manager’s problem into a pure adverse selection one. The conditions in the proposition then guarantee that, at each history, the manager prefers to follow a truthful and obedient strategy in the remaining periods rather than lying and then replicating the cash flows of the type being mimicked, irrespective of past effort, true and reported productivity.

4.2 Optimal policies

The next step is to use Condition (A) of Proposition 1 to derive an expression for the firm’s profits in terms only of the effort policy ξ and the period-1 compensation c_1 . This follows after observing that, given ξ and c_1 , the period-2 equilibrium compensation $c_2(\theta) = x_2(\theta, \pi(\theta))$ is uniquely determined by the need to provide the manager with a lifetime utility of monetary compensation equal to the level required by incentive compatibility, as given by (3). That is,

$$c_2(\theta) = w(W(\theta; \xi) + K - v(c_1(\theta_1))). \quad (7)$$

The following representation of the firm’s profits then follows from the result in Proposition 1.

Lemma 1 *Let $\langle \xi, c \rangle$ be implementable effort and compensation policies yielding an expected surplus of K to a manager with the lowest period-1 productivity $\underline{\theta}_1$. The firm’s expected profits under $\langle \xi, c \rangle$ are given by*

$$\mathbb{E}[U^P] = \mathbb{E}\left[\tilde{\theta}_1 + \xi_1(\tilde{\theta}_1) + \tilde{\theta}_2 + \xi_2(\tilde{\theta}) - c_1(\tilde{\theta}_1) - w\left(W(\tilde{\theta}; \xi) + K - v(c_1(\tilde{\theta}_1))\right)\right]. \quad (8)$$

Note that, when the manager is risk neutral ($v(y) = w(y)$ for all y), the result in Lemma 1 implies that the firm’s expected profits are equal to the entire surplus expected from the relationship, net of a term that corresponds to the expected surplus that the firm must leave to the manager and which depends only on the effort policy ξ :

$$\mathbb{E}[U^P] = \mathbb{E}\left[\begin{array}{l} \tilde{\theta}_1 + \xi_1(\tilde{\theta}_1) - \psi(\xi_1(\tilde{\theta}_1)) + \tilde{\theta}_2 + \xi_2(\tilde{\theta}) - \psi(\xi_2(\tilde{\theta})) \\ -\frac{1-F_1(\tilde{\theta}_1)}{f_1(\tilde{\theta}_1)} \left\{ \psi'(\xi_1(\tilde{\theta}_1)) + \gamma\psi'(\xi_2(\tilde{\theta}_1, \tilde{\theta}_2)) \right\} - K \end{array}\right]. \quad (9)$$

¹⁹Formally, let $\theta_2 = z(\theta_1, \varepsilon)$, where ε is a shock independent of θ_1 . The impulse response of θ_2 to θ_1 is the derivative of z with respect to θ_1 . In the case of a linear autoregressive process $\theta_2 = z(\theta_1, \varepsilon) = \gamma\theta_1 + \varepsilon$, so that the impulse response of θ_2 to θ_1 is equal to the persistence parameter γ . More generally, the impulse response of θ_2 to θ_1 is given by $I(\theta_1, \theta_2) \equiv \mathbb{E}\left[\frac{\partial z(\theta_1, \tilde{\varepsilon})}{\partial \theta_1} \Big| z(\theta_1, \tilde{\varepsilon}) = \theta_2\right]$.

The expression in (9) is what in the dynamic mechanism design literature (where payoffs are typically assumed to be quasilinear) is referred to as “dynamic virtual surplus”.

As one should expect, when instead the manager is risk averse, the firm’s payoff depends not only on the effort policy, but also on the way the compensation is spread over time. The value of the result in Lemma 1 comes from the fact that the choice over such compensation can be reduced to the choice over the period-1 compensation. This is because any two compensation schemes implementing the same effort policy ξ must give the manager the same utility of compensation not just in expectation, but ex-post, that is, at each productivity history $\theta = (\theta_1, \theta_2)$. This equivalence result (which is the dynamic analog in our non-quasilinear environment of the celebrated “revenue equivalence” for static quasilinear problems) plays an important role below in the characterization of the optimal policies.²⁰

We now consider the question of which implementable effort and compensation policies maximize the firm’s expected profits. As noted in the Introduction, the approach typically followed in the dynamic mechanism design literature to identify optimal policies is the following. First, consider a *relaxed program* that replaces all incentive-compatibility constraints with Condition (3) and all individual-rationality constraints with the constraint that $K = \mathbb{E}^{\tilde{\theta}|\underline{\theta}_1} [V(\tilde{\theta})] \geq 0$. Then choose policies (ξ_1, ξ_2, c_1) along with a scalar K to solve the unconstrained maximization of the firm’s profits as given by (8) and then let $c_2(\cdot)$ be given by (7).²¹ However, recall that, alone, Condition (3) is necessary but not sufficient for incentive compatibility. Furthermore, when the solution to the relaxed program yields policies prescribing a negative effort over a positive-measure set of types, satisfaction of the participation constraint for the lowest period-1 type $\underline{\theta}_1$ does not guarantee satisfaction of all other participation constraints. Therefore, one must typically identify auxiliary assumptions on the primitives of the problem guaranteeing that the effort and compensation policies $\langle \xi, c \rangle$ that solve the relaxed program are implementable.

The approach we follow here is different. Because the firm’s profits under any individually-rational and incentive-compatible contract must be consistent with the representation in (8), we use this expression to evaluate the performance of different contracts. However, not all policies (ξ_1, ξ_2, c_1) , coupled with c_2 as given in (7) for some $K \geq 0$, are implementable (in particular, this may be the case for those policies that maximize (8)). Hence, we do not aim at maximizing this expression directly. Instead, we use variational arguments to identify properties of optimal policies. More precisely, we first identify “admissible variations”. By this we mean perturbations to implementable policies such that the perturbed policies remain implementable (i.e., continue to satisfy the conditions of Proposition 1). For the candidate policies to be sustained under an optimal contract, it then must be the case that no admissible variation increases the firm’s profits, as expressed

²⁰See Pavan, Segal, and Toikka (2014) for a more general analysis of payoff-equivalence in dynamic settings.

²¹When the agent is risk neutral, the distribution of payments over time is irrelevant for the agent and hence (8) is independent of $c_1(\cdot)$. In this case, solving the relaxed program means finding an effort policy $\xi = (\xi_1, \xi_2)$ that maximizes (9) and then letting $c = (c_1, c_2)$ be any compensation policy that satisfies (3).

in (8).

Natural candidates for admissible variations are obtained by adding functions $\alpha(\theta_1)$ and $\beta(\theta)$ to the original effort policies $\xi_1(\theta_1)$ and $\xi_2(\theta)$, and then adjusting the compensation policy c so that payments continue to satisfy (3). While not all such variations are admissible (in particular, they need not yield effort policies satisfying the integral monotonicity constraints in (5)), it is easy to verify that, when the disutility of effort is quadratic, then adding non-negative constant functions $\alpha(\theta_1) = a > 0$ and $\beta(\theta) = b > 0$, all θ , to the original effort policies $\xi_1(\theta_1)$ and $\xi_2(\theta)$ and then adjusting the compensation policy c so that payments continue to satisfy (3) preserves all the constraints in Proposition 1. Furthermore, if the original policies $\langle \xi, c \rangle$ are such that the participation constraints bind at most only for the lowest period-1 type, θ_1 (which is always the case when the original effort policy ξ prescribes effort bounded away from zero from below at almost all histories), then we may also add negative constant functions, as long as $|a|$ and $|b|$ are small enough. The requirement that such perturbations do not increase the firm's expected profits then yields the following result. Let

$$\mathbb{E} \left[1 - \psi' \left(\xi_1^*(\tilde{\theta}_1) \right) w' \left(v \left(c_1^*(\tilde{\theta}_1) \right) \right) \right] \leq \mathbb{E} \left[\frac{\psi'' \left(\xi_1^*(\tilde{\theta}_1) \right)}{f_1(\tilde{\theta}_1)} \int_{\tilde{\theta}_1}^{\bar{\theta}_1} w' \left(v \left(c_1^*(r) \right) \right) f_1(r) dr \right], \quad (10)$$

$$\begin{aligned} \mathbb{E} \left[1 - \psi' \left(\xi_2^*(\tilde{\theta}) \right) w' \left(v \left(c_2^*(\tilde{\theta}) \right) \right) \right] &\leq \gamma \mathbb{E} \left[\frac{\psi'' \left(\xi_2^*(\tilde{\theta}) \right)}{f_1(\tilde{\theta}_1)} \int_{\tilde{\theta}_1}^{\bar{\theta}_1} w' \left(v \left(c_1^*(r) \right) \right) f_1(r) dr \right] \\ + \mathbb{E} \left[\frac{\psi'' \left(\xi_2^*(\tilde{\theta}) \right)}{f_2(\tilde{\theta}_2|\tilde{\theta}_1)} \int_{\tilde{\theta}_2}^{\bar{\theta}_2} \left\{ w' \left(v \left(c_2^*(\tilde{\theta}_1, r) \right) \right) - w' \left(v \left(c_1^*(\tilde{\theta}_1) \right) \right) \right\} f_2(r|\tilde{\theta}_1) dr \right], \text{ and} \end{aligned} \quad (11)$$

$$w' \left(v \left(c_1^*(\theta_1) \right) \right) = \mathbb{E}^{\tilde{\theta}_2|\theta_1} \left[w' \left(v \left(c_2^*(\theta_1, \tilde{\theta}_2) \right) \right) \right] \text{ all } \theta_1 \in \Theta_1. \quad (12)$$

Proposition 2 *Let $\langle \xi^*, c^* \rangle$ be effort and compensation policies sustained under an optimal contract. Then $\langle \xi^*, c^* \rangle$ must satisfy Conditions (10), (11) and (12).²² Furthermore, the inequalities in (10) and (11) must hold as equalities if $\psi'(\xi_1^*(\theta_1)) + \gamma \mathbb{E}^{\tilde{\theta}_1|\theta_1} [\psi'(\xi_2^*(\tilde{\theta}))]$ is bounded away from zero from below with probability one.*

Conditions (10) and (11) capture how the firm optimally solves the trade-off between increasing the manager's expected effort on the one hand and reducing the expected payments to the manager on the other. When the manager has preferences for consumption smoothing, his compensation must also be appropriately distributed over time according to Condition (12).

²²The effort policy implemented under any optimal contract is essentially unique, that is, unique, except over a zero-measure set of productivity histories. If v is strictly concave, the compensation policy implemented under any optimal contract is also essentially unique.

It is worth commenting on where our approach is similar to the one in the existing literature and where it departs. Condition (12) is obtained by considering perturbations to the compensation policy that leave the manager's payoff unchanged. In particular, we consider variations in period-1 compensation coupled with adjustments to the period-2 compensation chosen so that the total utility that the manager derives from his life-time compensation continues to satisfy (3). If the original policies $\langle \xi, c \rangle$ are implementable, so are the perturbed ones $\langle \xi, c' \rangle$. Therefore, under any optimal contract, such perturbations must not increase the firm's expected profits. For this to be the case, the proposed compensation scheme must satisfy Condition (12), which is the same inverse Euler condition

$$\frac{1}{v'(c_1^*(\theta_1))} = \mathbb{E}^{\tilde{\theta}_2|\theta_1} \left[\frac{1}{v'(c_2^*(\theta_1, \tilde{\theta}_2))} \right]$$

first identified by Rogerson (1985). The only novelty relative to Rogerson is that here the total utility from compensation is required to satisfy (3), which is necessary when the manager's productivity is his private information.

The point where our analysis departs from the rest of the literature is in the derivation of Conditions (10) and (11), which link the dynamics of effort to the dynamics of compensation, under optimal contracts. As mentioned above, these conditions are obtained by considering translations of the effort policy ξ that preserve implementability, i.e. that preserve Condition (B) in Proposition 1. Contrary to the perturbations of the compensation policy that lead to Condition (12), these perturbations necessarily change the manager's expected payoff, as one can readily see from (6). For these perturbations not to increase the firm's expected profits, it must be that the original policies satisfy Conditions (10) and (11) in the proposition.

Note that Conditions (10) and (11) hinge on our assumption that the disutility of effort is quadratic. As explained above, this assumption is what guarantees that translations of the effort policy ξ continue to satisfy Condition (B)(i) of Proposition 1. One might conjecture that our approach could be generalized to disutility functions that are not quadratic as follows: Rather than translating effort by a constant, one could translate the marginal disutility of effort. That is, one could consider the new effort policy given, for some $t \in \{1, 2\}$, by $\psi'(\xi_t^\eta(\theta)) = \psi'(\xi_t^*(\theta)) + \eta$ for η small, while letting $\xi_s^\eta(\theta) = \xi_s^*(\theta)$ for $s \neq t$. Unfortunately, the new effort policy ξ^η typically does not satisfy Condition B(i) of Proposition 1 (even though, by assumption, the original policy ξ^* does satisfy this condition). On the other hand, the assumptions that (a) productivity follows an AR(1) process and (b) there are only two periods are not essential for the result in Proposition 2. In fact, it is easy to verify that the same perturbations also preserve incentive compatibility in environments with more than two periods and richer stochastic processes. Euler conditions analogous to those in (10) and (11) can thus be obtained also for richer environments.

The next proposition uses an alternative class of perturbations that preserve not only incentive compatibility but also the manager's expected payoff conditional on his period-1 type θ_1 . This is

obtained by considering joint perturbations of ξ_1 and ξ_2 of opposite sign. The requirement that such perturbations not increase profits yields another Euler condition that links the effort and compensation policies across the two periods.

Proposition 3 *Let $\langle \xi^*, c^* \rangle$ be effort and compensation policies sustained under an optimal contract. The policies $\langle \xi^*, c^* \rangle$ must satisfy the following condition for almost all $\theta_1 \in \Theta_1$:*

$$\begin{aligned} \mathbb{E}^{\tilde{\theta}|\theta_1} \left[1 - \psi' \left(\xi_2^*(\tilde{\theta}) \right) w' \left(v(c_2^*(\tilde{\theta})) \right) \right] &= \gamma [1 - \psi' \left(\xi_1^*(\theta_1) \right) w' \left(v(c_1^*(\theta_1)) \right)] \\ + \mathbb{E}^{\tilde{\theta}|\theta_1} \left[\frac{\psi'' \left(\xi_2^*(\tilde{\theta}) \right)}{f_2(\tilde{\theta}_2|\tilde{\theta}_1)} \int_{\tilde{\theta}_2}^{\bar{\theta}_2} \left\{ w' \left(v(c_2^*(\tilde{\theta}_1, r)) \right) - w' \left(v(c_1^*(\tilde{\theta}_1)) \right) \right\} f_2(r|\tilde{\theta}_1) dr \right]. \end{aligned} \quad (13)$$

Interestingly, note that Conditions (10) and (11) in Proposition 2 above jointly imply that Condition (13) holds in expectation, but only when the inequalities in (10) and (11) hold as equalities. Thus, an advantage of the perturbations that lead to Proposition 3 is that they permit us to establish (13), without any restriction on the shape of the policies (in particular, these perturbations do not require that $\psi'(\xi_1^*(\theta_1)) + \gamma \mathbb{E}^{\tilde{\theta}|\theta_1} [\psi'(\xi_2^*(\tilde{\theta}))]$ is bounded away from zero from below with probability one). On the other hand, the new Euler condition (13) is established using the property that the productivity process is autoregressive. This assumption permits us to add a function $\alpha(\theta_1) = a q_1(\theta_1)$ to the period-1 effort policy $\xi_1(\theta_1)$ and then compensate the variation by deducting the function $\beta(\theta) = \frac{a}{\gamma} q(\theta_1)$ from the period-2 effort policy $\xi_2(\theta)$ preserving simultaneously the manager's period-1 expected payoff, as given by (6), and all the integral monotonicity conditions, as given by (5).

4.3 Dynamics of expected distortions

Our next objective is to understand how distortions in the provision of incentives for effort change with tenure under optimal contracts. First, we need a workable definition of the “distortions”.

Definition 2 (wedges) *For each $t = 1, 2$ and each $\theta = (\theta_1, \theta_2)$, the (local ex-post) distortions*

$$D_t(\theta) \equiv 1 - \psi'(\xi_t(\theta)) w'(v(c_t(\theta))) \quad (14)$$

in the provision of incentives under the (incentive-compatible) mechanism $\Omega = \langle \xi, x \rangle$ are given by the wedge between the marginal effect of higher effort on the firm's cash flows and its marginal effect on the compensation necessary to preserve the manager's utility constant.

Note that the formula of the wedge in (14) parallels the one in the new dynamic public finance literature; it captures the distortion in the provision of incentives due to the manager's private information (in a first-best world, the wedge would be equal to zero at all periods and across all states). Interestingly, note that the second term in (14) can also be related to a certain measure of "pay for performance". Consider payment schemes x where the payments in each period depend on

the history of observed cash flows only through the contemporaneous observations (that is, for all $t = 1, 2$, $x_t(\theta, \pi)$ depends on π only through π_t) and where each payment $x_t(\theta, \pi)$ is differentiable in the contemporaneous cash flows π_t . Recall that the dependence of the compensation scheme on the reported productivities is meant to capture changes to the compensation scheme proposed by the manager at the beginning of the period. It is then easy to see that any payment scheme with the above properties implementing the effort and consumption policies $\langle \xi, c \rangle$ must satisfy, for any θ ,

$$\begin{aligned} \frac{\partial x_1(\theta_1, \pi_1)}{\partial \pi_1} \Big|_{\pi_1 = \pi_1(\theta_1)} &= \psi'(\xi_1(\theta_1)) w'(v(c_1(\theta_1))) \\ \frac{\partial x_2(\theta, \pi_2)}{\partial \pi_2} \Big|_{\pi_2 = \pi_2(\theta)} &= \psi'(\xi_2(\theta)) w'(v(c_2(\theta))) \end{aligned}$$

with $\pi_t(\theta) = \theta_t + \xi_t(\theta)$, $t = 1, 2$. The second term in (14) thus coincides with the rate at which, under such schemes, the period- t compensation changes with the period- t cash flows, around the target level.²³ The above schemes, however, need not always implement the desired policies. Furthermore, even when they do, there typically exist other schemes that also implement the same policies. Hereafter, we thus focus on the dynamics of distortions under optimal contracts as opposed to the dynamics of specific compensation schemes. In particular, we are interested in how the dynamics of distortions are affected by the persistence of the manager's productivity (here captured by γ) and by the manager's degree of risk aversion.

Risk neutrality. We start with the following result.

Proposition 4 *Assume the manager is risk neutral (that is, v is the identity function). Then for all θ_1 ,*

$$\mathbb{E}^{\tilde{\theta}|\theta_1} \left[D_2(\tilde{\theta}) \right] = \gamma D_1(\theta_1). \quad (15)$$

The result follows directly from (13) (observe that, when v is the identity function, the last term in (13) is identically equal to zero). Hence, in absolute value, the average distortion is lower in period two than in period one when $\gamma < 1$ and is the same when $\gamma = 1$. Furthermore the sign of the average distortions is constant over time. Also note that, when the manager is risk neutral, distortions in

²³While differentiable schemes need not always implement the optimal policies, we conjecture that differentiable schemes can always implement policies which are virtually optimal. By this we mean the following. Let $\langle \xi^*, c^* \rangle$ be fully optimal policies. For any $\varepsilon > 0$ there exist policies $\langle \xi, c \rangle$ and a differentiable compensation scheme x such that the following are true: (i) the contract $\Omega \equiv \langle \xi, x \rangle$ is individually rational and incentive compatible for the manager; (ii) in each state θ , the compensation the manager receives under Ω is given by c ; and (iii) with probability one $\|(\xi(\theta), c(\theta)) - (\xi^*(\theta), c^*(\theta))\| \leq \varepsilon$. In other words, the firm can always implement policies arbitrarily close to the fully-optimal ones using differentiable schemes. Moreover, we conjecture that, when the manager is risk averse, if the policies $\langle \xi, c \rangle$ yield profits arbitrarily close to the ones under the fully optimal policies, then $\langle \xi, c \rangle$ must be arbitrarily close to $\langle \xi^*, c^* \rangle$ in the \mathcal{L}_1 norm. Virtually optimal policies can then be expected to inherit the same dynamic properties discussed below as the fully optimal policies. This is because the key properties discussed below refer to the *expectation* of $\psi'(\xi_t(\theta)) w'(v(c_t(\theta)))$, where the expectation is over all possible productivity histories.

the provision of incentives reduce to the wedge between the marginal effect of effort on the firm's cash flows and the marginal disutility of effort evaluated at the prescribed effort level $\xi_t^*(\theta)$. In this case, the Euler conditions (10) and (11) describe properties not only of the ex-ante distortions, but also of effort.

Proposition 5 *Assume the manager is risk neutral (that is, v is the identity function).*

(a) *Suppose that, on average, period-1 effort is distorted downwards relative to the first-best level (that is, $\mathbb{E}[\xi_1^*(\tilde{\theta}_1)] < 1 = e^{FB}$). Then expected effort is higher in the second period than in the first one when $\gamma < 1$ and is the same in the two periods when $\gamma = 1$.*

(b) *Suppose that there exists $\kappa > 0$ such that, under the optimal contract, $\mathbb{E}^{\tilde{\theta}|\theta_1}[V(\tilde{\theta})] \geq \kappa[\theta_1 - \underline{\theta}_1]$ all θ_1 . Then, on average, period-1 effort is distorted downwards relative to the first-best level.*

(c) *Suppose that $\psi'(\xi_1^*(\theta_1)) + \gamma\mathbb{E}^{\tilde{\theta}|\theta_1}[\psi'(\xi_2^*(\tilde{\theta}))]$ is bounded away from zero from below with probability one. Then participation constraints bind only for the lowest period-1 type.*

(d) *Either one of the following two sets of conditions guarantees that $\psi'(\xi_1^*(\theta_1)) + \gamma\mathbb{E}^{\tilde{\theta}|\theta_1}[\psi'(\xi_2^*(\tilde{\theta}))]$ is bounded away from zero from below with probability one: (i) $[1 - F_1(\theta_1)]/f_1(\theta_1)$ is non-increasing and strictly smaller than $(1+\gamma)/(1+\gamma^2)$; (ii) $\sup\{[1 - F_1(\theta_1)]/f_1(\theta_1)\} < (1+\gamma)/(1+\gamma^2) - (\bar{\theta}_1 - \underline{\theta}_1)$ and $F_2(\cdot|\cdot)$ satisfies the monotone-likelihood-ratio property (that is, for all $\theta'_1 \geq \theta_1$, $f_2(\theta_2|\theta'_1)/f_2(\theta_2|\theta_1)$ is non-decreasing in θ_2 over $\Theta_2(\theta'_1) \cap \Theta_2(\theta_1)$).*

The result in Part (a) follows from (15) by observing that the latter is equivalent to

$$\mathbb{E}^{\tilde{\theta}}[\xi_2^*(\tilde{\theta})] = 1 - \gamma + \gamma\mathbb{E}^{\tilde{\theta}_1}[\xi_1^*(\tilde{\theta})]$$

Hence if, on average, period-1 effort is distorted downwards relative to the first best, then period-2 effort is, on average, higher in the second period than in the first one when $\gamma < 1$ and is the same in both periods when $\gamma = 1$.

The result in part (b) in turn is established by noting that, when the condition holds, then the Euler Conditions (10) and (11) must hold as equalities and reduce to²⁴

$$\mathbb{E}^{\tilde{\theta}_1} \left[1 - \psi'(\xi_1^*(\tilde{\theta}_1)) \right] = \mathbb{E}^{\tilde{\theta}_1} \left[\frac{1 - F_1(\tilde{\theta}_1)}{f_1(\tilde{\theta}_1)} \right], \quad (16)$$

$$\mathbb{E}^{\tilde{\theta}} \left[1 - \psi'(\xi_2^*(\tilde{\theta})) \right] = \gamma\mathbb{E}^{\tilde{\theta}} \left[\frac{1 - F_1(\tilde{\theta}_1)}{f_1(\tilde{\theta}_1)} \right]. \quad (17)$$

Recall that the right-hand sides of (16) and (17) capture the effect of higher effort on the surplus that the firm must leave to the manager to induce him to reveal his productivity (this surplus is

²⁴That the Conditions (10) and (11) must hold as equalities follows from the fact that, in this case, perturbations such as those discussed before Proposition 2 with $a, b > -\kappa$ preserve both the integral monotonicity constraints of Proposition 1 as well as all participation constraints.

over and above the minimal compensation required to compensate the manager for his disutility of effort, as one can see by inspecting (6)). The reason why, in this case, the firm distorts downward the effort asked of those managers whose initial productivity is low is to reduce the rents it must leave to those managers whose initial productivity is high. When productivity is not fully persistent, these distortions are more effective in reducing managerial rents early in the relationship as opposed to later on. Distortions are therefore smaller at later dates, explaining why the expected power of incentives increases with tenure. The increase is most pronounced when productivity is least persistent. Indeed, as we approach the case where productivity is independent over time (i.e., when γ is close to zero), the expected effort the firm asks of each manager in the second period is close to the first-best level ($e^{FB} = 1$).

Next, consider Part (c). The result follows directly from the fact that the manager's equilibrium payoff must satisfy the envelope formula (6). When the expected net present value of effort discounted by impulse responses is bounded away from zero from below, then managers whose period-1 productivity is above the lowest possible level can always guarantee themselves a strictly positive payoff by mimicking lower types, implying that their participation constraints are necessarily slack.

Finally consider Part (d), which provides sufficient conditions for the optimal effort policy ξ^* to be such that $\psi'(\xi_1^*(\theta_1)) + \gamma \mathbb{E}^{\tilde{\theta}|\theta_1} [\psi'(\xi_2^*(\tilde{\theta}))]$ is bounded away from zero from below. The first condition requires that the hazard rate $f_1(\theta_1)/[1 - F(\theta_1)]$ of the period-1 distribution be non-decreasing (as typically assumed in the mechanism design literature) and strictly higher than $\frac{1+\gamma^2}{1+\gamma}$. In this case, the optimal effort policies are those that solve the "relaxed program" and are given by

$$\xi_1^R(\theta) = 1 - \frac{1 - F_1(\theta_1)}{f_1(\theta_1)} \quad (18)$$

$$\xi_2^R(\theta) = 1 - \gamma \frac{1 - F_1(\theta_1)}{f_1(\theta_1)}. \quad (19)$$

That these policies are implementable follows because $f_1(\theta_1)/[1 - F_1(\theta_1)]$ is non-decreasing, which guarantees that $\xi^R = (\xi_1^R, \xi_2^R)$ satisfies the monotonicity conditions B(i) and B(ii) of Proposition 1. In this case, effort increases over time towards its first-best level, not just in expectation, but along any productivity sequence.

When the hazard rate $f_1(\theta_1)/[1 - F(\theta_1)]$ of the period-1 distribution fails to be non-decreasing, however, the above policies may violate the integral monotonicity constraints in (5). When this happens, the result in Part d(ii) of the above proposition is particularly useful, for it implies that expected effort continues to increase over time as long as the inverse hazard rate of the period-1 distribution is small enough and the conditional distribution $F_2(\cdot|\cdot)$ satisfies the MLRP. Note that, when θ_t follows an autoregressive process, as assumed here, the latter requirement is a restriction on the distribution G of the innovation ε . That the conditional distribution $F_2(\cdot|\cdot)$ satisfies the MLRP guarantees that, under any optimal contract, period-2 effort is non-increasing in period-2 productivity θ_2 , for almost all θ_1 . As we show in the Appendix, this property, together with the

fact that the inverse hazard rate of the period-1 distribution is small enough, guarantee that, under the optimal policies, $\psi'(\xi_1^*(\theta_1)) + \gamma \mathbb{E}^{\tilde{\theta}|\theta_1} \left[\psi'(\xi_2^*(\tilde{\theta})) \right]$ continues to be bounded away from zero from below with probability one, which in turn implies that expected effort must increase over time. To illustrate, consider the following example.

Example 1 Suppose that θ_1 is drawn from an absolutely continuous distribution F_1 with support $[0, 1/4]$ and density

$$f_1(\theta_1) = \begin{cases} \frac{32}{5}(1 - 6\theta_1) & 0 \leq \theta_1 \leq \frac{1}{8} \\ \frac{32}{5}(6\theta_1 - \frac{1}{2}) & \frac{1}{8} < \theta_1 \leq \frac{1}{4} \end{cases}$$

In addition, suppose that $\theta_2 = \gamma\theta_1 + \tilde{\varepsilon}$ with $\gamma < 1$ and with ε drawn from a Uniform distribution with support $[-a, +a]$, for some $a \in \mathbb{R}_{++}$. Then the effort policy $\xi^R = (\xi_1^R, \xi_2^R)$ that solves the relaxed program (as given by (21) and (22) above) fails to be part of an optimal mechanism, for it violates the integral monotonicity constraints in (5). Nonetheless, one can verify that the conditions in Part d(ii) of Proposition 5 are satisfied. Hence expected effort necessarily increases over time.

Risk aversion. To understand how risk aversion affects the above conclusions, consider the following family of felicity functions. Let $(v_\rho)_{\rho \geq 0}$ be a collection of functions $v_\rho : \mathbb{R} \rightarrow \mathbb{R}$ with the following properties: (i) for each $\rho > 0$, v_ρ is surjective, continuously differentiable, increasing, and strictly concave, with $v_\rho(0) = 0$ and $v'_\rho(0) = 1$; (ii) v_0 is the identity function; (iii) v'_ρ converges to one, uniformly over c as $\rho \rightarrow 0$. Hence, $(v_\rho)_{\rho \geq 0}$ captures a family of utility functions such that ρ indexes the level of the manager's risk aversion and where the manager's preferences over compensation converge to the risk-neutral ones as $\rho \rightarrow 0$, uniformly over consumption levels.²⁵ Our key finding, however, is Proposition 7 below, which applies to arbitrary utility functions.

Proposition 6 Suppose there exist $a, b \in \mathbb{R}_{++}$ such that, for almost all $\theta_1 \in \Theta_1$, $\theta_2 \in \Theta_2(\theta_1)$, $a < f_1(\theta_1)$, $f_2(\theta_2|\theta_1) < b$. Fix the level of persistence $\gamma < 1$ of the manager's productivity, and assume that the manager's preferences over consumption in each period are represented by the function $v_\rho(\cdot)$, with the function family $(v_\rho)_{\rho \geq 0}$ satisfying the properties described above. Then there exists $\bar{\rho} > 0$ such that, for any $\rho \in [0, \bar{\rho}]$, under any optimal contract:

$$\left| \mathbb{E}^{\tilde{\theta}} \left[D_2(\tilde{\theta}) \right] \right| < \left| \mathbb{E}^{\tilde{\theta}_1} \left[D_1(\tilde{\theta}_1) \right] \right|$$

with $\text{sign} \left(\mathbb{E}^{\tilde{\theta}} \left[D_2(\tilde{\theta}) \right] \right) = \text{sign} \left(\mathbb{E}^{\tilde{\theta}_1} \left[D_1(\tilde{\theta}_1) \right] \right)$.

²⁵ An example is the family of utility functions $(v_\rho)_{\rho \geq 0}$ given by

$$v_\rho(x) = \begin{cases} x(1 - \rho^2) - \frac{\rho}{x + \sqrt{\rho}} + \rho^{3/2} & \text{if } x \geq 0 \\ x(1 + 2\rho - \rho^2) + \frac{\rho}{x - \sqrt{2 - \rho}} + \rho\sqrt{2 - \rho} & \text{if } x < 0 \end{cases}$$

for $\rho \in (0, 1)$ and by $v_\rho(x) = x$ for $\rho = 0$.

The result in the proposition thus establishes continuity of the dynamics of distortions in the degree of risk aversion, around the risk-neutral level. The role of the conditions in the proposition (the uniform bounds on the densities and the assumption of uniform convergence of the derivatives of the v_ρ functions to the derivative in the risk neutral case) is to guarantee that, if the dynamics of distortions for small degrees of risk aversion were the opposite of those in the risk neutral case, then one could construct implementable policies that would improve upon the optimal ones for $\rho = 0$. Note that the assumptions in the Theorem of Maximum are violated in our setting (in particular, the set of implementable policies need not be compact and continuous in ρ), which explains the need for the additional conditions in the proposition (as well as the length of the proof in the Appendix).

Importantly, also note that while the result in Proposition 6 focuses on the dynamics of distortions, the same properties apply to the expected effort levels. Precisely, assume that, when $\rho = 0$ (that is, in the risk neutral case), expected effort is higher at date 2 than at date 1 (recall that Proposition 5 provides primitive conditions for this to be the case). Then, under the conditions in the proposition, expected effort remains higher at date 2 than at date 1 also for $\rho > 0$ but small enough. Intuitively, this is because the dynamics of effort coincides with the dynamics of distortions when the manager is risk neutral, and are close to each other, when risk aversion is small. The result in the proposition extends to the family of iso-elastic felicity functions $v_\rho(c) = \frac{c^{1-\rho}-1}{1-\rho}$ for $\rho \geq 0$ often considered in the literature, as long as effort under the optimal policies is bounded. In this case, the restrictions on the densities can be dispensed with.

The levels of risk aversion for which the result in Proposition 6 holds (i.e., how large one can take $\bar{\rho}$) should be expected to depend on the persistence of initial productivity γ . For a fixed level of risk aversion, if γ is close to 1, i.e., if the initial productivity is highly persistent, then distortions increase, on average, over time, as stated in the next proposition. Thus, assuming period-1 distortions are positive, the above result about the dynamics of average distortions is *completely reversed*.

Proposition 7 *Fix the productivity distributions F_1 and G and assume that the felicity function v is strictly concave.*

(a) *Suppose that $\gamma = 1$. Then, under any optimal contract, for almost all θ_1 ,*²⁶

$$\mathbb{E}^{\tilde{\theta}|\theta_1} \left[D_2(\tilde{\theta}) \right] \geq D_1(\theta_1) \quad (20)$$

(b) *Suppose there exists $b \in \mathbb{R}_{++}$ such that, for all $\theta_1 \in \Theta_1$, $\theta_2 \in \Theta_2(\theta_1)$, $f_2(\theta_2|\theta_1) < b$. Suppose also that there exists $M \in \mathbb{R}_{++}$ and $\gamma' < 1$ such that, for all $\gamma \in [\gamma', 1]$, the optimal effort policy ξ^* is uniformly bounded (in absolute value) by M . Finally, suppose that, for $\gamma = 1$, the inequality in (20)*

²⁶The inequality is strict provided that $c_2^*(\theta_1, \cdot)$ varies with θ_2 over a subset of Θ_2 of positive probability under $F_2(\cdot|\theta_1)$. We expect that this condition holds in all but “knife-edge” cases. A sufficient condition, for instance, is that the hazard rate $\frac{f_1(\theta_1)}{1-F_1(\theta_1)}$ is increasing and that the manager’s degree of risk aversion is not too large.

is strict.²⁷ Then there exists $\bar{\gamma} \in [\gamma', 1)$ such that, for all $\gamma \in [\bar{\gamma}, 1]$, (20) holds as a strict inequality.

Consider Part (a), which assumes $\gamma = 1$. To ease the discussion, suppose that the effort asked by the firm in each period is strictly positive and that distortions are non-negative (note that the result in the proposition also applies to the case where the effort asked to certain types as well as the distortions are negative). Then note that, when the manager is risk averse, incentivizing high effort in period two is more costly for the firm. This is because a high effort requires a high sensitivity of pay to performance. This in turn exposes the manager to volatile compensation as a result of his own private uncertainty about period-2 productivity. Since the manager dislikes this volatility, he must be provided additional compensation by the firm. To save on managerial compensation, the firm then, on average, distorts period-2 incentives more than in period 1. To see this more formally, note that, when effort is bounded away from zero from below with probability one, the Euler conditions (10) and (11) in Proposition 2 must hold as equalities. It is then easy to see that the first two terms in the right-hand sides of these equations are identical. The key difference between the two periods is the third term in the right-hand side of (11) which is always positive and captures the effect of the volatility in the period-2 compensation on the surplus that the firm must give to the manager to induce him to participate. This volatility originates in the need to make period-2 compensation sensitive to period-2 performance to incentivize period-2 effort. Such volatility can be reduced by increasing the wedge in the second period. Under any optimal contract, distortions in the provision of incentives thus increase over time to reduce the manager's exposure to compensation risk.

One further way to understand why average distortions decline over time when the manager is risk averse and productivity is sufficiently persistent is as follows. Suppose that period-2 effort is restricted to depend only on period-1 productivity (that is, suppose both ξ_1 and ξ_2 depend only on θ_1). The manager's period-2 compensation can then be written as

$$w \left(\begin{array}{l} \psi(\xi_1(\theta_1)) + \psi(\xi_2(\theta_1)) + \int_{\theta_1}^{\theta_1} \{\psi'(\xi_1(s)) + \gamma\psi'(\xi_2(s))\} ds \\ + \left(\theta_2 - \mathbb{E}^{\tilde{\theta}_2|\theta_1}[\tilde{\theta}_2]\right) \psi'(\xi_2(\theta_1)) - v(c_1(\theta_1)) \end{array} \right).$$

It is then easy to see that the volatility of the period-2 compensation is increasing in the period-2 effort $\xi_2(\theta_1)$. When the manager is risk averse, w is strictly convex. By reducing ξ_2 , the firm then reduces the expected period-2 compensation, for any level of the period-1 productivity.²⁸ When

²⁷ Again, this follows if there exists a positive measure set of types θ_1 such that $c_2^*(\theta_1, \cdot)$ varies with θ_2 over a subset of Θ_2 of positive probability under $F_2(\cdot|\theta_1)$.

²⁸ If we restrict attention to effort policies that depend only on period-1 productivity, then the result in Proposition 7 applies not only to the dynamics of distortions but also to the dynamics of expected effort: i.e., expected effort declines over time under the assumptions of the proposition. When we do not impose this restriction, however, we have been unable to disentangle the effect of risk aversion on expected effort from its effect on the expected distortions. This appears difficult because of the need to control for the correlation between second-period compensation and second-period effort, conditional on the period-1 productivity.

$\gamma = 1$, distortions thus increase, on average, over time.

Now, consider Part (b) of the proposition. One should expect that whether distortions (on average) increase over time should depend on the persistence parameter γ . The result suggests that the average distortion increases over time when the persistence parameter γ is sufficiently close to 1. As noted in the Introduction, we obtain this result assuming that the optimal effort policies in these cases are uniformly (almost surely) bounded. While we believe only mild conditions (such as boundedness of the inverse hazard rate $\frac{1-F_1(\theta_1)}{f_1(\theta_1)}$) are needed to guarantee the existence of a uniform bound, we were unable to find an argument to guarantee it.

4.4 Further discussion of optimal policies

Conditions (10) and (11) are obtained by maximizing the firm's profits over all *implementable* policies. As noted above, an alternative (and more canonical) approach involves maximizing the firm's profits subject only to certain "local incentive constraints". In our environment, this amounts to maximizing (8) over all possible effort and compensation policies, thus ignoring the possibility that policies that maximize (8) need not be implementable by a contract which is individually rational and incentive compatible for the manager. One advantage of this alternative approach is that (when validated) it provides a characterization of the optimal policies at all possible histories. In our environment, this means that one can derive conditions analogous to (10) and (11) which hold ex-post, i.e. for each possible productivity history, as opposed to in expectation.

Proposition 8 *Suppose that the policies $(\xi_1^R, \xi_2^R, c_1^R)$ maximize (8) and let c_2^R be the period-2 compensation given by (7) for $K = 0$. Then, with probability one, the policies $\langle \xi^R, c^R \rangle = \langle (\xi_1^R, \xi_2^R), (c_1^R, c_2^R) \rangle$ must satisfy Condition (12) as well as the following conditions:*

$$1 - \psi'(\xi_1^R(\theta_1)) w'(v(c_1^R(\theta_1))) = \frac{\psi''(\xi_1^R(\theta_1))}{f_1(\theta_1)} \int_{\theta_1}^{\bar{\theta}_1} w'(v(c_1^R(r))) f_1(r) dr, \quad (21)$$

and

$$1 - \psi'(\xi_2^R(\theta)) w'(v(c_2^R(\theta))) = \gamma \frac{\psi''(\xi_2^R(\theta))}{f_1(\theta_1)} \int_{\theta_1}^{\bar{\theta}_1} w'_1(v(c_1^R(r))) f_1(r) dr \\ + \frac{\psi''(\xi_2^R(\theta))}{f_2(\theta_2|\theta_1)} \int_{\theta_2}^{\bar{\theta}_2} \{w'(v(c_2^R(\theta_1, r))) - w'(v(c_1^R(\theta_1)))\} f_2(r|\theta_1) dr. \quad (22)$$

The effort policy ξ^R is essentially unique. If v is strictly concave, then the compensation policy c^R is also essentially unique.

Observe that, when the manager is risk neutral, given that the disutility of effort is quadratic, conditions (21) and (22) reduce to Conditions (18) and (19) above. Recall from the discussion of Proposition 5 that these policies also solve the full program (and hence are sustained under optimal

contracts) when the inverse hazard rate of the period-1 distribution is non-increasing and strictly smaller than $(1 + \gamma)/(1 + \gamma^2)$. An implication is that managers whose initial productivity is high are asked to exert higher effort than those managers whose initial productivity is low. The reason for this finding relates once again to the effect of effort on managerial rents. When the inverse hazard rate of the period-1 distribution is non-increasing, the weight the firm assigns to rent extraction relative to efficiency (as captured by the inverse hazard rate $[1 - F_1(\theta_1)]/f_1(\theta_1)$) is smaller for higher types (recall that asking type θ_1 to exert more effort requires increasing the rent of all types $\theta'_1 > \theta_1$). As a result, the firm asks higher effort to those managers whose initial productivity is high. When it comes to the dynamics of effort, we then have the following comparison across types.

Corollary 1 *Suppose that the manager is risk neutral and that the inverse hazard rate of the period-1 distribution is (weakly) decreasing and strictly smaller than $(1 + \gamma)/(1 + \gamma^2)$. Then the increase in effort over time is larger for those managers whose initial productivity is low.*

The result reflects the fact that period-1 effort is more downward distorted for those managers whose initial productivity is low, implying that, over time, the correction is larger for those types. The result in the corollary thus yields another testable prediction: because productivity is positively correlated with performance and because, under risk neutrality, higher effort requires a higher sensitivity of pay to performance, the econometrician should expect to find a negative relationship between early performance and the increase in the sensitivity of pay to performance over time. Note that this prediction is not shared by the alternative theories (mentioned in the Introduction) which explain increases in the sensitivity of pay to performance over time.

Next, consider the case of a risk-averse manager. In this case, verifying that the policies $\langle \xi^R, c^R \rangle$ that solve the relaxed program are implementable is more difficult. This is typically done for numerical examples on a case-by-case basis. Below we illustrate the implications of Proposition 8 for the case of a risk-averse manager whose preferences over compensation are described by a CRRA felicity function with risk aversion parameter equal to $\eta \in [0, 1/2]$ (meaning that, for all $c \geq 0$, $v(c) = (c^{1-\eta} - 1)/(1 - \eta)$).²⁹ We consider the case where θ_1 is drawn from a uniform distribution over $[0, 1/2]$ and where the period-2 shock ε is drawn from a uniform distribution over $[-1/2, 1/2]$. We solve numerically for the policies $(\xi_1^R, \xi_2^R, c_1^R)$ that maximize (8) and then verify that these policies, along with the corresponding period-2 compensation policy c_2^R given by (7) for

²⁹Note that, contrary to what assumed in the model setup in the main text, this felicity function is not surjective and Lipschitz continuous over the entire real line. However, the numerical results do not hinge on the lack of these properties. In fact, under the optimal policies identified in the numerical analysis, consumption is bounded away from zero from below. One can then construct extensions \hat{v} of the assumed felicity function v such that (a) $\hat{v}(c) = v(c)$ for all $c > c_0 > 0$, (b) the numerical solutions under \hat{v} coincide with those under v , and (c) \hat{v} satisfies all the conditions in the model setup.

$K = 0$, satisfy all the implementability conditions of Proposition 1 (see the Supplementary Material for details). In the discussion below, we focus on how distortions under the optimal policies $(\xi_1^R, \xi_2^R, c_1^R)$ depend on the coefficient of productivity persistence γ , and on the coefficient of relative risk aversion, η .

Figure 1 below shows how period-1 distortions $D_1(\theta_1)$ and expected period-2 distortions $\mathbb{E}^{\tilde{\theta}|\theta_1} [D_2(\tilde{\theta})]$ vary with the initial productivity level θ_1 , when $\eta = 1/2$ and $\gamma = 1/2$ and $\gamma = 1$.

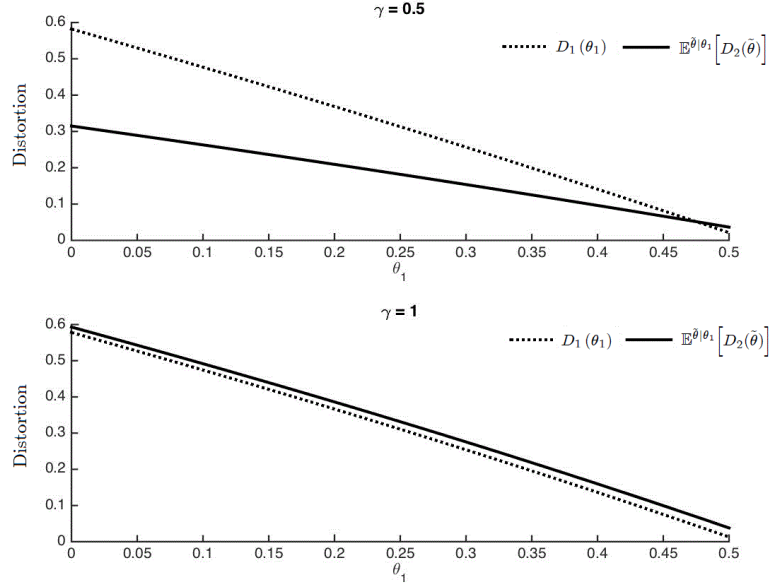


Figure 1: Optimal distortions: $\eta = 1/2$, $\gamma = 1/2$ and $\gamma = 1$.

When productivity is fully persistent (i.e., for $\gamma = 1$), for any θ_1 , period-1 distortions are higher than the expected period-2 distortions, thus illustrating the analytical finding in part (a) of Proposition 7. For $\gamma = 1/2$, instead, whether the expected period-2 distortions are higher or lower than the corresponding period-1 distortions depends on the initial productivity level. For high θ_1 , expected distortions increase over time, whereas the opposite is true for low θ_1 . These differences reflect the trade-off between reducing the managers' exposure to risk, which calls for reducing the sensitivity of pay to performance and effort at later periods, and reducing the managers' expected rents, which calls for higher distortions early on followed by smaller distortions later in the relationship. The effect of distortions on expected rents is similar across the two periods when either (i) productivity is fully persistent ($\gamma = 1$), or (ii) the initial productivity is high, in which case the effect of distortions on rents is negligible. In these cases, the firm optimally increases the expected distortion over time so as to reduce the risk the manager faces when it comes to his future compensation.

Next consider the effect of different levels of persistence on period-1 effort, $\xi_1^R(\theta_1)$ and expected period-2 effort, $\mathbb{E}^{\tilde{\theta}|\theta_1} [\xi_2^R(\tilde{\theta})]$, across different period-1 productivity levels. As Figures 2 below shows,

when $\eta = 1/2$ and $\gamma = 1/2$, expected period-2 effort is higher than period-1 effort for all types, except the highest. When, instead, $\eta = 1/2$ but $\gamma = 1$, expected period-2 effort is lower than period-1 effort, across all period-1 types. The figure also reveals that, for $\gamma = 1/2$, the increase in effort over time is larger for those managers whose initial productivity is the lowest, thus extending to the risk-averse case under examination here the result in Corollary 1 for risk neutral managers.

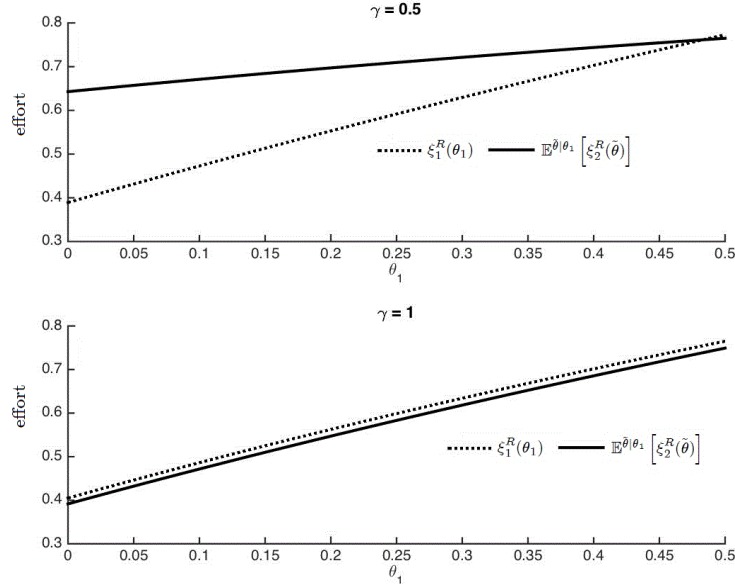


Figure 2: Optimal effort: $\eta = 1/2$, $\gamma = 1/2$ and $\gamma = 1$.

Next, recall that part (b) of Proposition 7 indicates that expected distortions should increase over time, across all θ_1 , for sufficiently large values of γ . How large γ has to be obviously depends on the degree of risk aversion η . In Figure 3 below, we continue to assume that $\eta = 1/2$ and plot the difference $\mathbb{E}^{\tilde{\theta}|\theta_1} [D_2(\tilde{\theta})] - D_1(\theta_1)$ between period-2 expected distortions and period-1 distortions across different θ_1 , for three different levels of persistence, $\gamma = .9$, $\gamma = .95$, and $\gamma = 1$. As proved in Proposition 7, part (a), when productivity is perfectly persistent ($\gamma = 1$), the difference is strictly positive for all θ_1 . When, instead, $\gamma = .95$ or $\gamma = .9$, the difference continues to be positive, but only for sufficiently high values of θ_1 . That, for low values of θ_1 , the difference is negative reflects the fact, for these types, period-1 effort is small. The firm can then afford to ask these types a higher period-2 effort without imposing them significant additional compensation risk.

The above results illustrate the effects on the dynamics of distortions of different persistence levels, for given level of managerial risk aversion ($\eta = 1/2$ in each of these figures). Figure 4 below, instead, fixes the level of persistence to $\gamma = .95$ and shows how the difference $\mathbb{E}^{\tilde{\theta}|\theta_1} [D_2(\tilde{\theta})] - D_1(\theta_1)$ between period-2 expected distortions and period-1 distortions for different values of θ_1 is affected by the degree of managerial risk aversion, η . As one may expect from the results in Propositions 6 and 7,

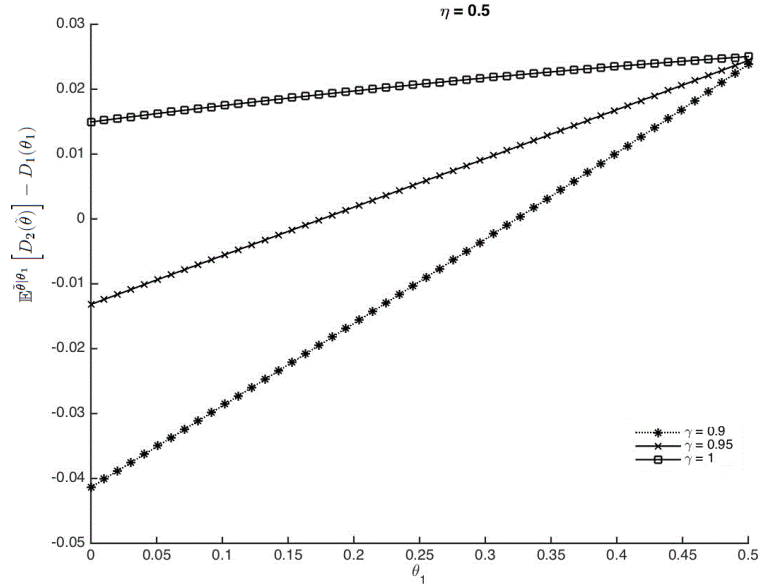


Figure 3: Differential between period-2 and period-1 distortions: $\eta = 1/2$, $\gamma = 0.9$, $\gamma = 0.95$ and $\gamma = 1$.

higher degrees of risk aversion imply a higher differential between period-2 expected distortions and period-1 distortions. In particular, Figure 4 reveals that, when $\eta = .05$ (that is, when the manager is close to being risk neutral) the expected period-2 distortions are smaller than period-1 distortions, for all but the very highest period-1 types, which is consistent with the findings in Proposition 6. For higher degrees of risk aversion, expected period-2 distortions are smaller than period-1 distortions over a smaller set of initial types θ_1 .

We conclude by showing how the unconditional difference between period-2 expected distortions and period-1 distortions is affected by different combinations of productivity persistence and risk aversion. In other words, we integrate over different values of θ_1 and show how the unconditional difference $\mathbb{E}^{\tilde{\theta}} [D_2(\tilde{\theta}) - D_1(\tilde{\theta}_1)]$ is affected by γ and η . The results are illustrated in Figure 5 below. As the figure reveals, average distortions are higher in period 2 than in period one for sufficiently high combinations of persistence and risk aversion, which is consistent with the analytical results of Proposition 7 (see also Figure 6, which helps interpreting the finding in Figure 5 by focusing on a restricted subset of parameter values).

We finally note that, in all numerical exercises, distortions are positive and effort is downward distorted and bounded away from zero in both periods.

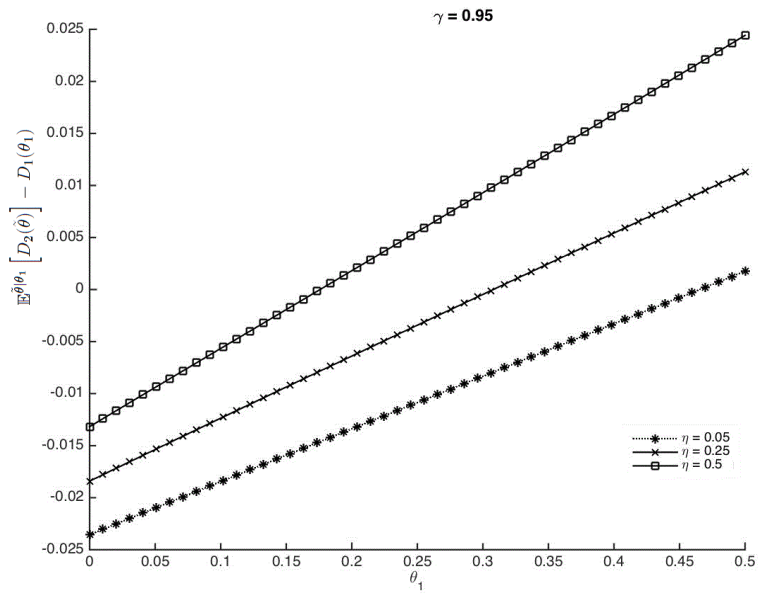


Figure 4: Differential between period-2 and period-1 distortions: $\gamma = 0.95$, $\eta = 0.05$, $\eta = 0.25$ and $\eta = 0.5$.

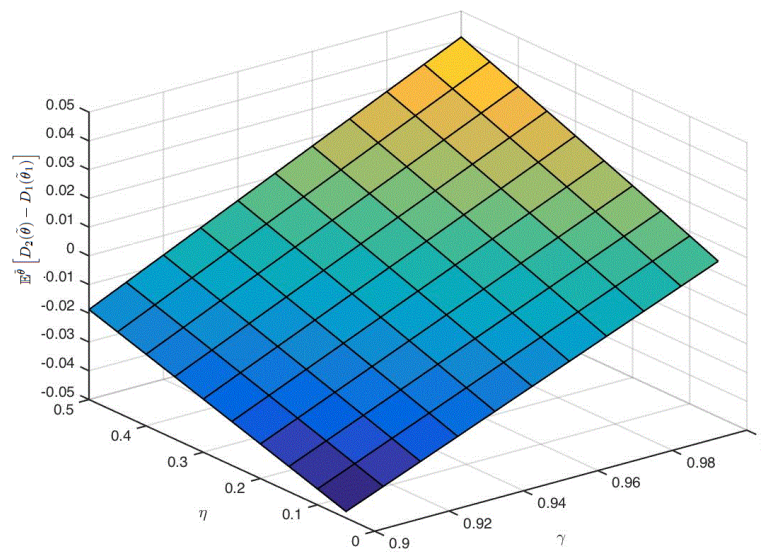


Figure 5: Unconditional difference between period-2 expected distortions and period-1 distortions.

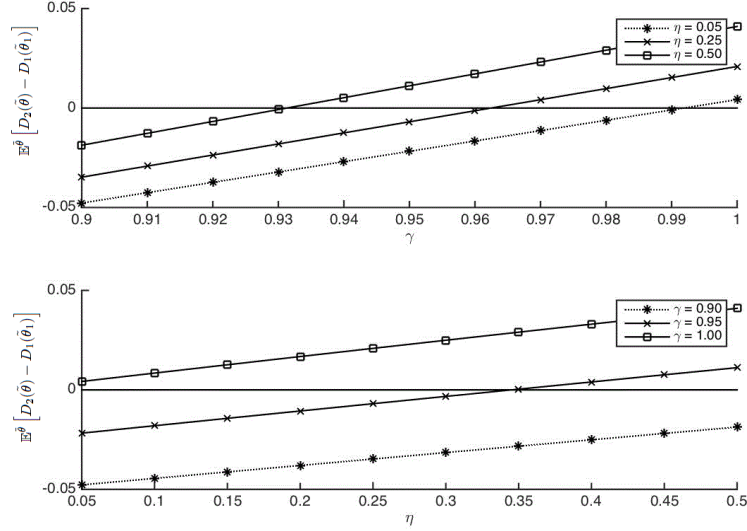


Figure 6: Unconditional difference between period-2 expected distortions and period-1 distortions.

5 Concluding discussion

We investigate the optimal dynamics of incentives for a manager whose ability to generate profits for the firm changes stochastically over time.

When the manager is risk neutral, we show that it is typically optimal for the firm to induce, on average, higher effort over time, thus reducing the expected distortions due to incomplete information. The above dynamics can be reversed under risk aversion. In future work, it would be interesting to calibrate the model so as to quantify the relevance of the effects identified in the paper and derive specific predictions about the combination of stocks, options, and fixed pay that implement the optimal dynamics of incentives.

We conclude with a few remarks about the applicability of the approach developed in the present paper (which involves tackling the full program directly) to richer specifications of the contracting problem. First, Euler inequalities like (10) and (11) in Proposition 2 can be obtained for settings with arbitrarily many periods and richer stochastic processes; these inequalities hold as equalities provided that optimal effort is not too small. When the manager is risk neutral, these equalities provide closed-form expressions for expected effort in each period (analogous to Equations (16) and (17) in the paper). Interestingly, these expressions can be obtained without any of the conditions typically imposed in the dynamic mechanism design literature (e.g., log-concavity of the period-1 distribution, monotonicity of the impulse responses of future types to the initial ones). This is because the predictions identified by this approach apply to the “average” dynamics, where the average is over all possible realizations of the type process, as opposed to ex-post. Equations relating

average distortions across periods, like the one in Proposition 3, can also be obtained for arbitrarily many periods. While no restriction on the shape of the effort policy is needed to establish such equations, the assumption that the process is autoregressive plays a role in the derivation of these equations and is more difficult to relax. This is because such equations are obtained by combining perturbations to the effort policy in one period with perturbations to the effort policy in other periods that preserve incentives, while also leaving the manager’s expected payoff unchanged. Identifying such multi-period perturbations for more general processes appears difficult.

Note also that, while we find the restriction to two periods helpful for drawing conclusions from the aforementioned Euler conditions, we expect our predictions for the dynamics of effort and expected distortions to extend to longer horizons. In particular, when the manager is risk neutral, and when the productivity process is imperfectly persistent (e.g., for a persistence parameter less than 1 in the autoregressive setting), we anticipate distortions to decrease on average over time under any optimal mechanism. Conversely, when the process is highly persistent (say close to a random walk), and when the manager is risk averse, then we expect distortions to increase over time. In this setting, the principal seeks to shield the manager from productivity risk later in the relationship when, from the perspective of the time of contracting, he faces the greatest uncertainty about his productivity. Shielding the manager from risk requires reducing the sensitivity of pay to performance, thus distorting the incentives for effort downwards relative to the first-best.

While our approach can be extended to longer relationships and richer stochastic processes, the assumption that the disutility of effort is quadratic is more difficult to relax. This assumption plays no role in the traditional approach (consisting in solving a “relaxed program” and then validating its solution). However, when tackling directly the “full program,” this assumption permits us to identify a simple class of perturbations that preserve incentive compatibility which can be used to arrive at the Euler equations in Propositions 2 and 3. In this respect, this assumption plays in our environment a role similar to that of the linearity of payoffs in Rochet and Choné’s (1998) analysis of multidimensional screening. There are two difficulties with more general effort disutility functions. The first one is in identifying appropriate perturbations of the effort policies that preserve incentive compatibility (see footnote 27). The second difficulty is in evaluating the marginal effects of such perturbations on the principal’s payoff. With more general effort disutility functions, the analogs of the Euler-type conditions that we used in the present paper appear less amenable to tractable analysis.

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Appendix

Proof of Proposition 1. Given the effort and compensation policies $\langle \xi, c \rangle$, let x be the compensation scheme defined, for each t , by

$$x_t(\theta, \pi) = \begin{cases} c_t(\theta) & \text{if } \pi_t = \pi_t(\theta) \\ -L_t(\theta) & \text{otherwise} \end{cases} \quad (23)$$

with $L_t(\theta) > 0$. It is easy to see that if the policies $\langle \xi, c \rangle$ are implementable, then there exists a compensation scheme x as given by (23) such that (i) the contract $\Omega = \langle \xi, x \rangle$ is incentive compatible and individually rational and (ii) the compensation that the manager receives on-path under x is the one prescribed by the policy c . Hereafter, we thus confine attention to contracts in which the compensation scheme is of the form given by (23).

Necessity. Recall that, by definition, if $\langle \xi, c \rangle$ are implementable, then there must exist a compensation contract x such that (i) the contract $\Omega = \langle \xi, x \rangle$ is incentive compatible and individually rational and (ii) the compensation that the manager receives on-path under x is the one prescribed by the policy c . In particular, incentive compatibility of $\Omega = \langle \xi, x \rangle$ requires that a manager of period-1 productivity θ_1 prefers to follow a truthful and obedient strategy in each period than lying about his period-1 productivity by reporting $\hat{\theta}_1$, then adjusting his period-1 effort so as to hide the lie (i.e., choosing effort $e_1 = \pi_1(\hat{\theta}_1) - \theta_1$ so as to generate the same cash flows as the type $\hat{\theta}_1$ being mimicked), and then lying again in period two by announcing, for any true period-2 type $\theta_2 = \gamma\theta_1 + \varepsilon$, a report $\hat{\theta}_2 = \gamma\hat{\theta}_1 + (\theta_2 - \gamma\theta_1)$, and finally adjusting his period-2 effort so as to hide again the new lie (i.e., choosing effort $e_2 = \pi_2(\hat{\theta}_1, \gamma\hat{\theta}_1 + \theta_2 - \gamma\theta_1) - \theta_2$ so as to generate the same cash flows as those expected from someone whose true type history is $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2)$, with $\hat{\theta}_2 = \gamma\hat{\theta}_1 + (\theta_2 - \gamma\theta_1)$). Note that, for any $\theta_1, \hat{\theta}_1 \in \Theta_1$, the expected payoff

$$U_1(\theta_1, \hat{\theta}_1) \equiv \mathbb{E}^{\tilde{\varepsilon}} \left[\begin{array}{c} c_1(\hat{\theta}_1) + c_2(\hat{\theta}_1, \gamma\hat{\theta}_1 + \tilde{\varepsilon}) \\ -\psi \left(\pi_1(\hat{\theta}_1) - \theta_1 \right) - \psi \left(\pi_2(\hat{\theta}_1, \gamma\hat{\theta}_1 + \tilde{\varepsilon}) - \gamma\theta_1 - \tilde{\varepsilon} \right) \end{array} \right]$$

that the manager obtains from following such a strategy corresponds to the one that the manager obtains by lying in period 1 and then reporting the true shock ε truthfully in period two (and choosing effort in each period so as to generate the same cash flows as the ones expected from the reported types).

Likewise, let

$$U_2(\theta, \hat{\theta}) \equiv c_1(\hat{\theta}_1) + c_2(\hat{\theta}) - \psi \left(\pi_1(\hat{\theta}_1) - \theta_1 \right) - \psi \left(\pi_2(\hat{\theta}) - \theta_2 \right)$$

denote the ex-post payoff of a manager whose true productivity history is $\theta = (\theta_1, \theta_2)$, who reported $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2)$, and whose effort choices are made to perfectly hide the lies in each period.

The Lemma below establishes monotonicity properties of the equilibrium-cash flows which in turn will permit us to establish that, for any $(\theta_1, \hat{\theta}_1)$, $U_1(\theta_1, \hat{\theta}_1)$ is differentiable and equi-Lipschitz

continuous in θ_1 and that, for any $(\theta, \hat{\theta})$, $U_2(\theta, \hat{\theta})$ is differentiable and equi-Lipschitz continuous in θ_2 .

Lemma 2 *Suppose that the policies $\langle \xi, c \rangle$ are implementable and let $\langle \pi_t(\theta) \rangle$ be the equilibrium cash flows under such policies. Then necessarily $\pi_1(\theta_1) + \gamma \mathbb{E}^{\tilde{\varepsilon}}[\pi_2(\theta_1, \gamma\theta_1 + \tilde{\varepsilon})]$ is non-decreasing in θ_1 and, for any θ_1 , $\pi_2(\theta_1, \theta_2)$ is non-decreasing in θ_2 .*

Proof. That, for any θ_1 , $\pi_2(\theta_1, \theta_2)$ is non-decreasing in θ_2 follows directly from the fact that the manager's flow payoff $c_t - \psi(\pi_t - \theta_t)$ satisfies the increasing difference property with respect to (π_t, θ_t) . That $\pi_1(\theta_1) + \gamma \mathbb{E}^{\tilde{\varepsilon}}[\pi_2(\theta_1, \gamma\theta_1 + \tilde{\varepsilon})]$ must be non-decreasing in θ_1 can be seen by combining any pair of IC constraints

$$U_1(\theta_1, \theta_1) \geq U_1(\theta_1, \hat{\theta}_1) \text{ and } U_1(\hat{\theta}_1, \hat{\theta}_1) \geq U_1(\hat{\theta}_1, \theta_1).$$

From these constraints one obtains that

$$\begin{aligned} & \psi(\pi_1(\theta_1) - \hat{\theta}_1) + \mathbb{E}^{\tilde{\varepsilon}} \left[\psi(\pi_2(\theta_1, \gamma\theta_1 + \tilde{\varepsilon}) - \gamma\hat{\theta}_1 - \tilde{\varepsilon}) \right] \\ & - \left\{ \psi(\pi_1(\theta_1) - \theta_1) + \mathbb{E}^{\tilde{\varepsilon}} \left[\psi(\pi_2(\theta_1, \gamma\theta_1 + \tilde{\varepsilon}) - \gamma\theta_1 - \tilde{\varepsilon}) \right] \right\} \\ \geq & \psi(\pi_1(\hat{\theta}_1) - \hat{\theta}_1) + \mathbb{E}^{\tilde{\varepsilon}} \left[\psi(\pi_2(\hat{\theta}_1, \gamma\hat{\theta}_1 + \tilde{\varepsilon}) - \gamma\hat{\theta}_1 - \tilde{\varepsilon}) \right] \\ & - \left\{ \psi(\pi_1(\hat{\theta}_1) - \theta_1) + \mathbb{E}^{\tilde{\varepsilon}} \left[\psi(\pi_2(\hat{\theta}_1, \gamma\hat{\theta}_1 + \tilde{\varepsilon}) - \gamma\theta_1 - \tilde{\varepsilon}) \right] \right\}. \end{aligned}$$

From the fundamental theorem of calculus, we can rewrite the above inequality as

$$\begin{aligned} & \int_{\hat{\theta}_1}^{\theta_1} \psi'(\pi_1(\theta_1) - y) + \gamma \mathbb{E}^{\tilde{\varepsilon}} \left[\psi'(\pi_2(\theta_1, \gamma\theta_1 + \tilde{\varepsilon}) - \gamma y - \tilde{\varepsilon}) \right] dy \\ \geq & \int_{\hat{\theta}_1}^{\theta_1} \psi'(\pi_1(\hat{\theta}_1) - y) + \gamma \mathbb{E}^{\tilde{\varepsilon}} \left[\psi'(\pi_2(\hat{\theta}_1, \gamma\hat{\theta}_1 + \tilde{\varepsilon}) - \gamma y - \tilde{\varepsilon}) \right] dy. \end{aligned}$$

Using the fact that ψ is quadratic, we can in turn rewrite the above inequality as

$$\left(\theta_1 - \hat{\theta}_1 \right) \left(\pi_1(\theta_1) - \pi_1(\hat{\theta}_1) + \gamma \mathbb{E}^{\tilde{\varepsilon}} \left[\pi_2(\theta_1, \gamma\theta_1 + \tilde{\varepsilon}) - \pi_2(\hat{\theta}_1, \gamma\hat{\theta}_1 + \tilde{\varepsilon}) \right] \right) \geq 0,$$

which holds only if $\pi_1(\theta_1) + \gamma \mathbb{E}^{\tilde{\varepsilon}}[\pi_2(\theta_1, \gamma\theta_1 + \tilde{\varepsilon})]$ is non-decreasing in θ_1 . ■

The monotonicities of the cash flows in the Lemma, along with the compactness of Θ_1 and Θ_2 , in turn imply that (a), for any $(\theta, \hat{\theta})$, $U_2(\theta, \hat{\theta})$ is differentiable and Lipschitz continuous in θ_2 with Lipschitz constant

$$M_2(\hat{\theta}_1) = \max_{\hat{\theta}_2 \in \Theta_2} \{ |\pi_2(\hat{\theta}_1, \hat{\theta}_2)| \} + \max \{ |\gamma \underline{\theta}_1 + \underline{\varepsilon}|, |\gamma \bar{\theta}_1 + \bar{\varepsilon}| \}$$

uniform across $(\theta_2, \hat{\theta}_2)$ and (b) for any $(\theta_1, \hat{\theta}_1)$, $U_1(\theta_1, \hat{\theta}_1)$ is differentiable and Lipschitz continuous in θ_1 with Lipschitz constant

$$M_1 = \max_{\hat{\theta}_1 \in \Theta_1} \{ |\pi_1(\hat{\theta}_1) + \gamma \mathbb{E}^{\tilde{\varepsilon}}[\pi_2(\hat{\theta}_1, \gamma\hat{\theta}_1 + \tilde{\varepsilon})]| \} + \max \{ |\underline{\theta}_1|, |\bar{\theta}_1| \} + \gamma \max \{ |\gamma \underline{\theta}_1 + \underline{\varepsilon}|, |\gamma \bar{\theta}_1 + \bar{\varepsilon}| \}$$

uniform across $(\theta_1, \hat{\theta}_1)$. Using results from the recent dynamic mechanism design literature, one can then show that the following conditions are necessary for incentive compatibility: (1) for any (θ_1, θ_2) , the manager's ex-post equilibrium payoff satisfies

$$V(\theta_1, \theta_2) = V(\theta_1, \underline{\theta}_2) + \int_{\underline{\theta}_2}^{\theta_2} \psi'(\xi_2(\theta_1, s)) ds; \quad (24)$$

and (2) for each θ_1 , the expectation of the equilibrium payoff satisfies (6), where $V(\theta_1, \theta_2) = U_2((\theta_1, \theta_2), (\theta_1, \theta_2))$ and $V_1(\theta_1) \equiv \mathbb{E}^{\tilde{\theta}|\theta_1}[V(\tilde{\theta})] = U_1(\theta_1, \theta_1)$. Note that Condition (24) is analogous to the static condition in Laffont and Tirole (1986). The necessity of (6), instead, follows from adapting to the environment under examination the result in Theorem 1 in Pavan, Segal, and Toikka (2014).

Combining (24) with (6), we then obtain that, under any contract that is individually rational and incentive compatible, the equilibrium utility that each manager derives from his lifetime compensation must satisfy Condition (3) for all $\theta = (\theta_1, \theta_2)$, with $K = \mathbb{E}^{\tilde{\theta}|\theta_1}[V(\tilde{\theta})] \geq 0$ satisfying Condition (4). This establishes the necessity of Condition (A) in the proposition. The necessity of Condition (B)(ii) follows directly from Lemma 2 above.

Finally, to see that Condition (B)(i) is also necessary, let $\Omega = \langle \xi, x \rangle$ be any contract implementing the effort and compensation policies $\langle \xi, c \rangle$. Then let $V^\Omega(\theta_1, \hat{\theta}_1)$ be the payoff that, under such a contract, a manager whose period-1 productivity is θ_1 obtains when he reports $\hat{\theta}_1$, then chooses period-1 effort $e_1 = \pi_1(\hat{\theta}_1) - \theta_1$ optimally so as to attain the target $\pi_1(\hat{\theta}_1)$, and then behaves optimally in period 2 (which means following a truthful and obedient strategy³⁰). Then observe that

$$\begin{aligned} V^\Omega(\theta_1, \hat{\theta}_1) &= V^\Omega(\hat{\theta}_1, \hat{\theta}_1) + \psi(\xi_1(\hat{\theta}_1)) - \psi(\xi_1(\hat{\theta}_1) + \hat{\theta}_1 - \theta_1) \\ &\quad + \mathbb{E}^{\tilde{\theta}_2|\theta_1} \left[\int_{\underline{\theta}_2}^{\tilde{\theta}_2} \psi'(\xi_2(\hat{\theta}_1, s)) ds \right] - \mathbb{E}^{\tilde{\theta}_2|\hat{\theta}_1} \left[\int_{\underline{\theta}_2}^{\tilde{\theta}_2} \psi'(\xi_2(\hat{\theta}_1, s)) ds \right] \\ &= V^\Omega(\hat{\theta}_1, \hat{\theta}_1) + \int_{\hat{\theta}_1}^{\theta_1} \left\{ \psi'(\xi_1(\hat{\theta}_1) + \hat{\theta}_1 - s) + \gamma \mathbb{E}^{\tilde{\theta}_2|s} [\psi'(\xi_2(\hat{\theta}_1, \tilde{\theta}_2))] \right\} ds. \end{aligned} \quad (25)$$

Because the policies $\langle \xi, c \rangle$ implemented under the contract Ω must satisfy (3), we have that

$$V^\Omega(\theta_1, \theta_1) = V^\Omega(\hat{\theta}_1, \hat{\theta}_1) + \int_{\hat{\theta}_1}^{\theta_1} \left\{ \psi'(\xi_1(s)) + \mathbb{E}^{\tilde{\theta}_2|s} [\psi'(\xi_2(s, \tilde{\theta}_2))] \right\} ds. \quad (26)$$

A necessary condition for incentive compatibility is that $V^\Omega(\theta_1, \hat{\theta}_1) \leq V^\Omega(\theta_1, \theta_1)$ for all $\theta_1, \hat{\theta}_1 \in \Theta_1$. Using (25) and (26), the latter condition is equivalent to the integral-monotonicity condition (5).

Sufficiency. Suppose that the policies $\langle \xi, c \rangle$ satisfy all the conditions in the proposition. Consider the scheme x given by (23) with $L_t(\theta) > 0$ for each t . Because, for any t , any $\hat{\theta}$, $\pi_t(\hat{\theta})$ is finite

³⁰Note that the optimality of truthful and obedient behavior at *all* period-2 histories follows from the combination of the fact that the environment is Markov along with the fact that, for any θ_1 , (a) the equilibrium cash flows $\pi_2(\theta_1, \cdot)$ are nondecreasing in θ_2 , and (b) the effort and compensation policies satisfy the envelope condition (24), which is implied by (3). The result then follows directly from Laffont and Tirole (1986).

and because Θ_t is bounded, it is easy to see that there exist finite penalties $L_t(\theta)$ such that, faced with the above scheme, for any history of reports $\hat{\theta}$ and any history of true types θ , the period- t optimal choice of effort is $\pi_t(\hat{\theta}) - \theta_t$, irrespective of past effort choices. It is also easy to see that, under such a scheme, the manager finds it optimal to follow a truthful and obedient strategy in the second period, irrespective of his period-1 true and reported type, and irrespective of the effort exerted in period one (the arguments for this result are similar to those in Laffont and Tirole (1986) and hence omitted).

To establish the result, it then suffices to show that, under the proposed scheme, a manager of period-1 productivity θ_1 prefers to follow a truthful and obedient strategy in both periods than lying by reporting $\hat{\theta}_1 \neq \theta_1$ in period one, then optimally choosing effort $e_1 = \pi_1(\hat{\theta}_1) - \theta_1$ so as to attain the target $\pi_1(\hat{\theta}_1)$, and then following a truthful and obedient strategy in period two. Under the scheme x , the payoff that the manager expects from a truthful and obedient strategy in both periods is given by (26), whereas the payoff that he expects by lying in period one and then following the optimal behavior described above is the one in (25). That $V^\Omega(\theta_1, \hat{\theta}_1) \leq V^\Omega(\theta_1, \theta_1)$ for all $\theta_1, \hat{\theta}_1 \in \Theta_1$ then follows from the fact that the policies $\langle \xi, c \rangle$ satisfy the integral-monotonicity condition (5). Q.E.D.

Proof of Proposition 2. The proof is in two steps. Step 1 identifies a family of perturbations that preserve incentive compatibility and then uses this family to identify necessary conditions for the proposed effort and compensation policies $\langle \xi^*, c^* \rangle$ to be sustained under an optimal contract. Step 2 establishes the uniqueness of $\langle \xi^*, c^* \rangle$.

Step 1 (Euler Equations). We want to establish that Conditions (10), (11), and (12) are necessary optimality conditions for the policies ξ^* and c^* . To see this, consider the perturbed effort policy $\xi = (\xi_1^*(\cdot) + a, \xi_2^*(\cdot) + b)$ for some constants $a, b \in \mathbb{R}_+$. Then consider the perturbed compensation policy c given by $c_1(\theta_1) = c_1^*(\theta_1)$ and $c_2(\theta) = w(W(\theta; \xi) + K - v(c_1^*(\theta_1)))$ all θ , where $K = \mathbb{E}^{\tilde{\theta}|\theta_1}[V(\tilde{\theta})]$ is the the lowest period-1 type's expected payoff under the original policies $\langle \xi^*, c^* \rangle$.

It is easy to see that, if the policies $\langle \xi^*, c^* \rangle$ are implementable (which, by virtue of Proposition 1, means that they satisfy the conditions in Proposition 1), then so are the perturbed policies $\langle \xi, c \rangle$.

Now consider the firm's expected profits under the perturbed policies. For the original policies $\langle \xi^*, c^* \rangle$ to be optimal, the expected profits must be maximized at $a = b = 0$. Using (8), we have that the right-hand derivative of the firm's expected profits with respect to a , evaluated at $a = b = 0$ is non-positive only if the policies ξ^* and c^* satisfy Condition (10) (to see this, it suffices to take the right-hand derivative of $\mathbb{E}[U^P]$ with respect to a and then integrate by parts). Likewise, the right-hand derivative of $\mathbb{E}[U^P]$ with respect to b , evaluated at $a = b = 0$, is non-positive only if the policies satisfy (11).

Next observe that, when the policy ξ^* is such that $\psi'(\xi_1^*(\theta_1)) + \gamma \mathbb{E}^{\tilde{\theta}|\theta_1}[\psi'(\xi_2^*(\tilde{\theta}))]$ is (almost surely) bounded away from zero from below, then perturbations like the ones described above but with $a, b \in \mathbb{R}_-$, with $|a|$ and $|b|$ small to guarantee that the resulting policies continue to satisfy $\psi'(\xi_1(\theta_1)) +$

$\gamma \mathbb{E}^{\tilde{\theta}|\theta_1} [\psi'(\xi_2(\tilde{\theta}))] \geq 0$ for (almost) all θ_1 , also yield implementable policies (that such perturbations preserve integral monotonicity is obvious; the role of the bound on $\psi'(\xi_1^*(\theta_1)) + \gamma \mathbb{E}^{\tilde{\theta}|\theta_1} [\psi'(\xi_2^*(\tilde{\theta}))]$ is to guarantee that such perturbations leave the participation constraints of all types satisfied). Also note that, in this case, the left-hand derivatives of the firm's expected profits with respect to a and b , evaluated at $a = b = 0$ coincide with their right-hand analogs. Optimality of the policies $\langle \xi^*, c^* \rangle$ then requires that such derivatives vanish at $a = b = 0$, which is the case only if the inequalities in (10) and (11) hold as equalities.

The argument for the necessity of (12) is similar. Fix the effort policy ξ^* and consider a perturbation of the period-1 compensation policy so that the new policy satisfies $v(c_1(\theta_1)) = v(c_1^*(\theta_1)) + a\eta(\theta_1)$ for a scalar a and some measurable function $\eta(\cdot)$. In other words, $c_1(\theta_1) = w(v(c_1^*(\theta_1)) + a\eta(\theta_1))$. Then adjust the period-2 compensation so that $c_2(\theta) = w(W(\theta; \xi^*) + K - v(c_1(\theta_1)))$ all θ . It is easy to see that the pair of policies $\langle \xi^*, c \rangle$ continues to be implementable. The firm's expected profits under the perturbed policies are

$$\mathbb{E}[U^P] = \mathbb{E} \left[\begin{array}{l} \tilde{\theta}_1 + \xi_1^*(\tilde{\theta}_1) + \tilde{\theta}_2 + \xi_2^*(\tilde{\theta}) \\ -w(v(c_1^*(\tilde{\theta}_1)) + a\eta(\tilde{\theta}_1)) - w(W(\tilde{\theta}; \xi^*) - v(c_1^*(\tilde{\theta}_1)) - a\eta(\theta_1)) \end{array} \right].$$

Optimality of c^* then requires that the derivative of this expression with respect to a vanishes at $a = 0$ for all measurable functions η . This is the case only if Condition (12) holds.

Step 2 (Uniqueness of the optimal policies). We first show that the optimal effort policy is essentially unique (i.e., unique up to a zero-measure set of productivity histories). Suppose, towards a contradiction, that there exist two pairs of optimal (implementable) policies, $\langle \xi^\#, c^\# \rangle$ and $\langle \xi^{\#\#}, c^{\#\#} \rangle$ respectively, and that $\xi^\#$ and $\xi^{\#\#}$ prescribe different effort levels over a set of productivity histories of strictly positive probability measure. Pick $\alpha \in (0, 1)$ and let $\xi^\alpha \equiv \alpha\xi^\# + (1 - \alpha)\xi^{\#\#}$ be the policy defined by $\xi_t^\alpha(\theta) = \alpha\xi_t^\#(\theta) + (1 - \alpha)\xi_t^{\#\#}(\theta)$ for all θ and $t = 1, 2$. Then let c_1^α be the policy defined, for all θ , by $c_1^\alpha(\theta) \equiv w(\alpha v(c_1^\#(\theta)) + (1 - \alpha)v(c_1^{\#\#}(\theta)))$. Finally, let c_2^α be the policy defined, for all θ , by $c_2^\alpha(\theta) \equiv w(W(\theta; \xi^\alpha) + \alpha K^\# + (1 - \alpha)K^{\#\#} - v(c_1^\alpha(\theta_1)))$, where $K^\#$ and $K^{\#\#}$ denote type θ_1 's expected payoff under the policies $\langle \xi^\#, c^\# \rangle$ and $\langle \xi^{\#\#}, c^{\#\#} \rangle$, respectively. Note that the new policies $\langle \xi^\alpha, c^\alpha \rangle$ are implementable (to see this, note that they satisfy the conditions of Proposition 1).

Next, note that (8) is strictly concave in the effort policy ξ (recognizing that the policy ξ enters (8) also through $W(\theta; \xi)$, as defined in (2)) and weakly concave in K and $v(c_1)$.³¹ This means that the firm's expected profits $\mathbb{E}[U^P]$ under the new policies $\langle \xi^\alpha, c^\alpha \rangle$ are strictly higher than under either $\langle \xi^\#, c^\# \rangle$ or $\langle \xi^{\#\#}, c^{\#\#} \rangle$, contradicting the optimality of these policies.

³¹By *strict* concavity we mean with respect to the equivalence classes of functions which are equivalent if they are equal almost surely.

Now consider the uniqueness of the compensation policy. Suppose that v is strictly concave and let $\langle \xi^\#, c^\# \rangle$ and $\langle \xi^{\#\#}, c^{\#\#} \rangle$ be two pairs of implementable policies such that $c_1^\#(\theta_1) \neq c_1^{\#\#}(\theta_1)$ over a set of positive probability measure. Then consider the policies $\langle \xi^\alpha, c^\alpha \rangle$ constructed above. Note that such policies yield strictly higher profits than both $\langle \xi^\#, c^\# \rangle$ and $\langle \xi^{\#\#}, c^{\#\#} \rangle$, irrespective of whether or not $\xi^\# \neq \xi^{\#\#}$. This in turn implies that, when v is strictly concave, the optimal compensation policy is also (essentially) unique. Q.E.D.

Proof of Proposition 3. We establish the result by considering perturbations of the effort policy given by

$$\xi_1^\#(\theta_1) = \xi_1^*(\theta_1) + a q(\theta_1) \quad \text{and} \quad \xi_2^\#(\theta) = \xi_2^*(\theta) - \frac{a}{\gamma} q(\theta_1)$$

for some measurable function $q(\theta_1)$. Note that such perturbations leave period-1 expected payoffs unchanged and are implementable. Optimality of the policies $\langle \xi^*, c^* \rangle$ then requires that the derivative of the firm's expected payoff with respect to a , evaluated at $a = 0$ must vanish, for all possible $q(\cdot)$. This leads to the following new Euler equation, for each θ_1 :

$$0 = 1 - \frac{\psi''(\xi_1^*(\theta_1))}{f_1(\theta_1)} \int_{\theta_1}^{\bar{\theta}_1} w'(v(c_1^*(r))) f_1(r) dr - \psi'(\xi_1^*(\theta_1)) w'(v(c_1^*(\theta_1))) \\ - \frac{1}{\gamma} \left(1 - \gamma \mathbb{E}^{\tilde{\theta}|\theta_1} \left[\frac{\psi''(\xi_2^*(\tilde{\theta}))}{f_1(\tilde{\theta}_1)} \int_{\tilde{\theta}_1}^{\bar{\theta}_1} w'(v(c_1^*(r))) f_1(r) dr \right] - \mathbb{E}^{\tilde{\theta}|\theta_1} \left[\psi'(\xi_2^*(\tilde{\theta})) w'(v(c_2^*(\tilde{\theta}))) \right] \right) \\ - \mathbb{E}^{\tilde{\theta}|\theta_1} \left[\frac{\psi''(\xi_2^*(\tilde{\theta}))}{f_2(\tilde{\theta}_2|\tilde{\theta}_1)} \int_{\tilde{\theta}_2}^{\bar{\theta}_2} \left\{ w'(v(c_2^*(\tilde{\theta}_1, r))) - w'(v(c_1^*(\tilde{\theta}_1))) \right\} f_2(r|\tilde{\theta}_1) dr \right] \right).$$

which is equivalent to (13) in the proposition. Q.E.D.

Proof of Proposition 4. The result follows from the arguments in the main text.

Proof of Proposition 5. The proof for Parts (a), (b), and (c) follows from the arguments in the main text. Thus consider Part (d)(i). In this case, the optimal effort policies are those that solve the relaxed program, as given in (18) and (19); that is, $\xi_1^*(\theta_1) = 1 - \frac{1-F_1(\theta_1)}{f_1(\theta_1)}$ and $\xi_2^*(\theta) = 1 - \gamma \frac{1-F_1(\theta_1)}{f_1(\theta_1)}$. Hence,

$$\begin{aligned} \psi'(\xi_1^*(\theta_1)) + \gamma \mathbb{E}^{\tilde{\theta}|\theta_1} \left[\psi'(\xi_2^*(\tilde{\theta})) \right] &= \xi_1^*(\theta_1) + \gamma \mathbb{E}^{\tilde{\theta}|\theta_1} \left[\xi_2^*(\tilde{\theta}) \right] \\ &= 1 - \frac{1 - F_1(\theta_1)}{f_1(\theta_1)} + \gamma \left(1 - \gamma \frac{1 - F_1(\theta_1)}{f_1(\theta_1)} \right) \\ &\geq 1 - \frac{1 - F_1(\underline{\theta}_1)}{f_1(\underline{\theta}_1)} + \gamma \left(1 - \gamma \frac{1 - F_1(\underline{\theta}_1)}{f_1(\underline{\theta}_1)} \right) > 0 \end{aligned}$$

where the first inequality follows from the assumption that $[1 - F_1(\theta_1)]/f_1(\theta_1)$ is non-increasing, and where the second inequality from the assumption that $[1 - F_1(\theta_1)]/f_1(\theta_1) < (1 + \gamma)/(1 + \gamma^2)$, for all θ_1 .

Next consider Part (d)(ii). Suppose that $\sup\{[1 - F_1(\theta_1)]/f_1(\theta_1)\} < (1 + \gamma)/(1 + \gamma^2) - (\bar{\theta}_1 - \underline{\theta}_1)$ and $F_2(\cdot|\cdot)$ satisfies the monotone-likelihood-ratio property (that is, for all $\theta'_1 \geq \theta_1$, $f_2(\theta_2|\theta'_1)/f_2(\theta_2|\theta_1)$ is non-decreasing in θ_2 over $\Theta_2(\theta'_1) \cap \Theta_2(\theta_1)$). We want to show that $\psi'(\xi_1^*(\theta_1)) + \gamma \mathbb{E}^{\tilde{\theta}|\theta_1} [\psi'(\xi_2^*(\tilde{\theta}))]$ is bounded away from zero from below with probability one. We proceed in two steps. Step 1 establishes four lemmas that jointly imply that it is without loss of optimality to restrict attention to effort policies such that, for all θ_1 , $\xi_2(\theta_1, \cdot)$ is non-increasing in θ_2 . Step 2 then use this property to establish that, under the conditions in part d(ii) in the proposition, if $\langle \xi^*, c^* \rangle$ is such that $\psi'(\xi_1^*(\theta_1)) + \gamma \mathbb{E}^{\tilde{\theta}|\theta_1} [\psi'(\xi_2^*(\tilde{\theta}))]$ fails to be bounded away from zero from below with probability one, then there exists another pair of policies $\langle \hat{\xi}, \hat{c} \rangle$ that is also implementable and yields strictly higher profits, thus contradicting the optimality of $\langle \xi^*, c^* \rangle$.

Step 1. Before proceeding, note that we can restrict attention to effort policies $\xi = \langle \xi_1, \xi_2 \rangle$ such that

$$\xi_2(\theta_1, \theta_2) = \begin{cases} \xi_2(\theta_1, \underline{\theta}_2(\theta_1)) + (\underline{\theta}_2(\theta_1) - \theta_2) & \text{if } \theta_2 < \underline{\theta}_2(\theta_1) \\ \xi_2(\theta_1, \bar{\theta}_2(\theta_1)) - (\theta_2 - \bar{\theta}_2(\theta_1)) & \text{if } \theta_2 > \bar{\theta}_2(\theta_1), \end{cases}$$

and where $\xi_2(\theta_1, \underline{\theta}_2(\theta_1)) = \lim_{\theta_2 \downarrow \underline{\theta}_2(\theta_1)} \xi_2(\theta_1, \theta_2)$ and $\xi_2(\theta_1, \bar{\theta}_2(\theta_1)) = \lim_{\theta_2 \uparrow \bar{\theta}_2(\theta_1)} \xi_2(\theta_1, \theta_2)$. To see this, consider any implementable effort and consumption policy $\langle \xi, c \rangle$, and consider the policy $\langle \hat{\xi}, \hat{c} \rangle$ which specifies, for all θ_1 , $\hat{\xi}_1(\theta_1) = \xi_1(\theta_1)$, $\hat{\xi}_2(\theta) = \xi_2(\theta)$ for $\theta_2 \in (\underline{\theta}_2(\theta_1), \bar{\theta}_2(\theta_1))$, $\hat{\xi}_2(\theta_1, \underline{\theta}_2(\theta_1)) = \lim_{\theta_2 \downarrow \underline{\theta}_2(\theta_1)} \xi_2(\theta_1, \theta_2)$, $\hat{\xi}_2(\theta_1, \bar{\theta}_2(\theta_1)) = \lim_{\theta_2 \uparrow \bar{\theta}_2(\theta_1)} \xi_2(\theta_1, \theta_2)$, and

$$\hat{\xi}_2(\theta_1, \theta_2) = \begin{cases} \hat{\xi}_2(\theta_1, \underline{\theta}_2(\theta_1)) + (\underline{\theta}_2(\theta_1) - \theta_2) & \text{if } \theta_2 < \underline{\theta}_2(\theta_1) \\ \hat{\xi}_2(\theta_1, \bar{\theta}_2(\theta_1)) - (\theta_2 - \bar{\theta}_2(\theta_1)) & \text{if } \theta_2 > \bar{\theta}_2(\theta_1). \end{cases}$$

Finally, let $\hat{c}_1 = c_1$ and then let \hat{c}_2 be determined by (7), using $\hat{\xi}$ and \hat{c}_1 . The policy $\langle \hat{\xi}, \hat{c} \rangle$ is implementable and generates the same payoff for the firm as the original policy $\langle \xi, c \rangle$ (implementability can be checked with respect to the conditions in Proposition 1). Note hence that, for the policies we consider, $\xi_2(\theta_1, \cdot)$ is decreasing in θ_2 for $\theta_2 \leq \underline{\theta}_2(\theta_1)$ and $\theta_2 \geq \bar{\theta}_2(\theta_1)$. It is thus left to show that we can restrict attention to policies such that, for each θ_1 , $\xi_2(\theta_1, \cdot)$ is non-increasing in θ_2 over $\theta_2 \in \Theta_2(\theta_1)$.

We next establish the following result.

Lemma 3 *Fix any θ_1 . Consider any function $h : \Theta_2(\theta_1) \rightarrow \mathbb{R}$ which is continuous at the endpoints $\underline{\theta}_2(\theta_1)$ and $\bar{\theta}_2(\theta_1)$ and such that $h(\theta_2) + \theta_2$ is non-decreasing on $\Theta_2(\theta_1)$. Suppose that h fails to be non-increasing on $\Theta_2(\theta_1)$; in particular, there exist $\theta'_2, \theta''_2 \in \Theta_2(\theta_1)$, $\theta'_2 < \theta''_2$ such that $h(\theta'_2) < h(\theta''_2)$. Take any $\bar{h} \in (h(\theta'_2), h(\theta''_2))$. There exists $\theta_2^\#, \theta_2^{\#\#} \in \Theta_2$, with $\theta_2^\# \leq \theta_2^{\#\#}$, and $\delta^\#, \delta^{\#\#} > 0$ such that (i) for all $\theta_2 \in (\theta_2^\# - \delta^\#, \theta_2^\#)$, $h(\theta_2) < \bar{h}$, and for all $\theta_2 \in (\theta_2^{\#\#}, \theta_2^{\#\#} + \delta^{\#\#})$, $h(\theta_2) > \bar{h}$; and (ii) $\lim_{\theta_2 \downarrow \theta_2^\#} h(\theta_2) \geq \bar{h}$ and $\lim_{\theta_2 \uparrow \theta_2^{\#\#}} h(\theta_2) \leq \bar{h}$.*

Proof. It suffices to take

$$\theta_2^\# = \sup \left\{ \theta_2 : h(\tilde{\theta}_2) < \bar{h} \forall \tilde{\theta}_2 \in (\theta'_2, \theta_2) \right\} \text{ and } \theta_2^{\#\#} = \inf \left\{ \theta_2 : h(\tilde{\theta}_2) > \bar{h} \forall \tilde{\theta}_2 \in (\theta_2, \theta''_2) \right\},$$

and then let $\delta^\# = \theta_2^\# - \theta_2' + \frac{\bar{h} - h(\theta_2')}{2}$ and $\delta^{\#\#} = \theta_2'' - \theta_2^{\#\#} + \frac{h(\theta_2'') - \bar{h}}{2}$. That $\lim_{\theta_2 \downarrow \theta_2^\#} h(\theta_2)$ exists follows from the fact that $\lim_{\theta_2 \downarrow \theta_2^\#} (h(\theta_2) + \theta_2)$ exists, which in turn follows from the fact that $h(\theta_2) + \theta_2$ is non-decreasing. That $\lim_{\theta_2 \downarrow \theta_2^\#} h(\theta_2) \geq \bar{h}$ follows from the fact that, if this was not true, then there would exist $\hat{\theta}_2 > \theta_2^\#$ such that $h(\hat{\theta}_2) < \bar{h}$ for all $\tilde{\theta}_2 \in (\theta_2', \hat{\theta}_2)$, thus contradicting the definition of $\theta_2^\#$. The proof of the fact that $\lim_{\theta_2 \uparrow \theta_2^{\#\#}} h(\theta_2)$ exists and is such that $\lim_{\theta_2 \uparrow \theta_2^{\#\#}} h(\theta_2) \leq \bar{h}$ follows from similar arguments. ■

Lemma 4 Fix θ_1 and let F_2 be a distribution on $\Theta_2(\theta_1)$. Consider any function $h : \Theta_2(\theta_1) \rightarrow \mathbb{R}$ which is continuous at the end-points $\underline{\theta}_2(\theta_1)$ and $\bar{\theta}_2(\theta_1)$ and such that $h(\theta_2) + \theta_2$ is non-decreasing on $\Theta_2(\theta_1)$, and suppose that h fails to be non-increasing. Take $\bar{h}, \theta_2^\#, \theta_2^{\#\#}, \delta^\#, \delta^{\#\#}$ as defined in the previous lemma. For arbitrary $\delta^* \in (0, \delta^\#)$ and $\delta^{**} \in (0, \delta^{\#\#})$, define the function $h^*(\theta_2; \delta^*, \delta^{**})$ by $h^*(\theta_2; \delta^*, \delta^{**}) = \bar{h}$ for $\theta_2 \in [\theta_2^\# - \delta^*, \theta_2^\#] \cup [\theta_2^{\#\#}, \theta_2^{\#\#} + \delta^{**}]$, and $h^*(\theta_2; \delta^*, \delta^{**}) = h(\theta_2)$ otherwise. (i) For any $\delta^* \in (0, \delta^\#)$ and $\delta^{**} \in (0, \delta^{\#\#})$, $h^*(\theta_2; \delta^*, \delta^{**}) + \theta_2$ is non-decreasing over $\Theta_2(\theta_1)$. (ii) There exist $\delta^* \in (0, \delta^\#)$ and $\delta^{**} \in (0, \delta^{\#\#})$ such that $\mathbb{E}^{F_2} [h^*(\tilde{\theta}_2; \delta^*, \delta^{**})] = \mathbb{E}^{F_2} [h(\tilde{\theta}_2)]$, where the expectation is taken under F_2 ; equivalently,

$$\mathbb{E}^{F_2} [h(\tilde{\theta}_2) | \tilde{\theta}_2 \in [\theta_2^\# - \delta^*, \theta_2^\#] \cup [\theta_2^{\#\#}, \theta_2^{\#\#} + \delta^{**}]] = \bar{h}.$$

Moreover, we can choose δ^* and δ^{**} so that $\theta_2^\# - \delta^*, \theta_2^{\#\#} + \delta^{**} \in \Theta_2(\theta_1)$.

Proof. To prove (i) one need only to verify that $h^*(\theta_2; \delta^*, \delta^{**}) + \theta_2$ is non-decreasing at $\theta_2^\#$ and at $\theta_2^{\#\#}$. By the definition of $\theta_2^\#, \theta_2^{\#\#}$ and of the h^* function, it is easy to see that

$$\lim_{\theta_2 \downarrow \theta_2^\#} (h^*(\theta_2; \delta^*, \delta^{**}) + \theta_2) \geq \theta_2^\# + \bar{h} = \theta_2^\# + h^*(\theta_2^\#; \delta^*, \delta^{**}).$$

and

$$\lim_{\theta_2 \uparrow \theta_2^{\#\#}} (h^*(\theta_2; \delta^*, \delta^{**}) + \theta_2) \leq \bar{h} + \theta_2^{\#\#} = \theta_2^{\#\#} + h^*(\theta_2^{\#\#}; \delta^*, \delta^{**}).$$

The proof for part (ii) follows from the fact that $h^*(\theta_2; \delta^*, \delta^{**}) = h(\theta_2)$ for all $\theta_2 \notin [\theta_2^\# - \delta^*, \theta_2^\#] \cup [\theta_2^{\#\#}, \theta_2^{\#\#} + \delta^{**}]$, $h^*(\theta_2; \delta^*, \delta^{**}) > h(\theta_2)$ for all $\theta_2 \in [\theta_2^\# - \delta^*, \theta_2^\#]$, and $h^*(\theta_2; \delta^*, \delta^{**}) < h(\theta_2)$ for all $\theta_2 \in [\theta_2^{\#\#}, \theta_2^{\#\#} + \delta^{**}]$. ■

Now consider any θ_1 for which $\xi_2(\theta_1, \theta_2) + \theta_2$ is non-decreasing in θ_2 , as required by incentive compatibility, but for which $\xi_2(\theta_1, \cdot)$ fails to be non-increasing over $\Theta_2(\theta_1)$. Letting $h(\theta_2) = \xi_2(\theta_1, \theta_2)$, the above two lemmas permit us to establish the following result.

Lemma 5 Consider any θ_1 for which $\xi_2(\theta_1, \theta_2) + \theta_2$ is non-decreasing in θ_2 , but for which $\xi_2(\theta_1, \cdot)$ fails to be non-increasing over $\Theta_2(\theta_1)$. Suppose that the distribution $F_2(\cdot | \cdot)$ satisfies the MLRP (that is, for all $\theta_1' \geq \theta_1$, $f_2(\theta_2 | \theta_1') / f_2(\theta_2 | \theta_1)$ is non-decreasing in θ_2 over $\Theta_2(\theta_1') \cap \Theta_2(\theta_1)$). Then there exists a function $\hat{\xi}_2(\theta_1, \cdot) : \Theta_2 \rightarrow \mathbb{R}$ such that (a) $\mathbb{E}^{\tilde{\theta}_2 | \theta_1} [\hat{\xi}_2(\theta_1, \tilde{\theta}_2)] = \mathbb{E}^{\tilde{\theta}_2 | \theta_1} [\xi_2(\theta_1, \tilde{\theta}_2)]$, (b)

$\hat{\xi}_2(\theta_1, \theta_2) + \theta_2$ is non-decreasing in θ_2 , (c) for all $s < \theta_1$, $\mathbb{E}^{\tilde{\theta}_2|s} [\hat{\xi}_2(\theta_1, \tilde{\theta}_2)] \geq \mathbb{E}^{\tilde{\theta}_2|s} [\xi_2(\theta_1, \tilde{\theta}_2)]$, while, for all $s > \theta_1$, $\mathbb{E}^{\tilde{\theta}_2|s} [\hat{\xi}_2(\theta_1, \tilde{\theta}_2)] \leq \mathbb{E}^{\tilde{\theta}_2|s} [\xi_2(\theta_1, \tilde{\theta}_2)]$, and (d)

$$- \int_{\Theta_2} \left(\frac{\hat{\xi}_2(\theta_1, \theta_2)^2}{2} - \frac{\xi_2(\theta_1, \theta_2)^2}{2} \right) dF_2(\theta_2|\theta_1) > 0. \quad (27)$$

Proof. Take any θ_1 for which the properties in the lemma hold. Let $h(\theta_2) = \xi_2(\theta_1, \theta_2)$, and $\hat{\xi}_2(\theta_1, \theta_2) = h^*(\theta_2; \delta^*, \delta^{**})$, where the function h^* (and hence the values \bar{h} , $\theta_2^{\#\#}$, $\theta_2^{\#\#}$, δ^* and δ^{**}) are as defined as in the previous lemma. That $\hat{\xi}_2(\theta_1, \theta_2)$ satisfies properties (a) and (b) follows directly from the above two lemmas.

Next consider property (c). Consider $s > \theta_1$ (the proof for $s < \theta_1$ is symmetric and hence omitted). We have that

$$\begin{aligned} & \mathbb{E}^{\tilde{\theta}_2|s} [\hat{\xi}_2(\theta_1, \tilde{\theta}_2)] - \mathbb{E}^{\tilde{\theta}_2|s} [\xi_2(\theta_1, \tilde{\theta}_2)] = \int_{(\theta_2^{\#} - \delta^*, \theta_2^{\#})} \left(\hat{\xi}_2(\theta_1, \theta_2) - \xi_2(\theta_1, \theta_2) \right) f_2(\theta_2|s) d\theta_2 \\ & + \int_{(\theta_2^{\#\#}, \theta_2^{\#\#} + \delta^{**})} \left(\hat{\xi}_2(\theta_1, \theta_2) - \xi_2(\theta_1, \theta_2) \right) f_2(\theta_2|s) d\theta_2 \\ & = \int_{(\theta_2^{\#} - \delta^*, \theta_2^{\#})} \left(\hat{\xi}_2(\theta_1, \theta_2) - \xi_2(\theta_1, \theta_2) \right) \frac{f_2(\theta_2|s)}{f_2(\theta_2|\theta_1)} f_2(\theta_2|\theta_1) d\theta_2 \\ & + \int_{(\theta_2^{\#\#}, \theta_2^{\#\#} + \delta^{**})} \left(\hat{\xi}_2(\theta_1, \theta_2) - \xi_2(\theta_1, \theta_2) \right) \frac{f_2(\theta_2|s)}{f_2(\theta_2|\theta_1)} f_2(\theta_2|\theta_1) d\theta_2 \\ & \leq \int_{(\theta_2^{\#} - \delta^*, \theta_2^{\#})} \left(\hat{\xi}_2(\theta_1, \theta_2) - \xi_2(\theta_1, \theta_2) \right) \frac{f_2(\theta_2^{\#}|s)}{f_2(\theta_2^{\#}|\theta_1)} f_2(\theta_2|\theta_1) d\theta_2 \\ & + \int_{(\theta_2^{\#\#}, \theta_2^{\#\#} + \delta^{**})} \left(\hat{\xi}_2(\theta_1, \theta_2) - \xi_2(\theta_1, \theta_2) \right) \frac{f_2(\theta_2^{\#}|s)}{f_2(\theta_2^{\#}|\theta_1)} f_2(\theta_2|\theta_1) d\theta_2 \\ & = \frac{f_2(\theta_2^{\#}|s)}{f_2(\theta_2^{\#}|\theta_1)} \left(\mathbb{E}^{\tilde{\theta}_2|\theta_1} [\hat{\xi}_2(\theta_1, \tilde{\theta}_2)] - \mathbb{E}^{\tilde{\theta}_2|\theta_1} [\xi_2(\theta_1, \tilde{\theta}_2)] \right) = 0. \end{aligned}$$

where, for the inequality, we used the fact that, by construction of the function $\hat{\xi}_2(\theta_1, \cdot)$, $\hat{\xi}_2(\theta_1, \theta_2) \geq \xi_2(\theta_1, \theta_2)$ for $\theta_2 \in (\theta_2^{\#} - \delta^*, \theta_2^{\#})$ and $\hat{\xi}_2(\theta_1, \theta_2) \leq \xi_2(\theta_1, \theta_2)$ for $\theta_2 \in (\theta_2^{\#\#}, \theta_2^{\#\#} + \delta^{**})$, along with the fact that $f_2(\theta_2|s)/f_2(\theta_2|\theta_1)$ is increasing in θ_2 by the MLRP, while, for the equality, we used the property in part (a).

Finally, property (d) follows from Jensen's inequality after noting that, for any

$$\theta_2 \in S \equiv \left(\theta_2^{\#} - \delta^*, \theta_2^{\#} \right) \cup \left(\theta_2^{\#\#}, \theta_2^{\#\#} + \delta^{**} \right),$$

$\hat{\xi}_2(\theta_1, \theta_2) = \mathbb{E}^{\tilde{\theta}_2|\theta_1} [\xi_2(\theta_1, \tilde{\theta}_2)|\theta_2 \in S]$, while $\hat{\xi}_2(\theta_1, \theta_2) = \xi_2(\theta_1, \theta_2)$ for $\theta_2 \notin S$. ■

We then have the following result.

Lemma 6 *Suppose that $F_2(\cdot|\cdot)$ satisfies the MLRP. For any pair of implementable policies $\langle \xi, c \rangle$ such that $\xi_2(\theta_1, \cdot)$ fails to be non-increasing in θ_2 (on $\Theta_2(\theta_1)$) over a positive measure subset of Θ_1 , there exist implementable policies $\langle \hat{\xi}, \hat{c} \rangle$ such that the principal's expected profits under $\langle \hat{\xi}, \hat{c} \rangle$ are strictly higher than under $\langle \xi, c \rangle$.*

Proof. Let $\hat{\xi}_1 = \xi_1$. For any θ_1 such that $\xi_2(\theta_1, \cdot)$ is non-increasing in θ_2 , let $\hat{\xi}_2(\theta_1, \cdot) = \xi_2(\theta_1, \cdot)$, while for any θ_1 for which $\xi_2(\theta_1, \cdot)$ fails to be non-increasing in θ_2 (on $\Theta_2(\theta_1)$), take $\hat{\xi}_2(\theta_1, \theta_2)$ as in the previous lemma. Then let $\hat{c}_1(\cdot) = c_1(\cdot)$ and for any θ , let $\hat{c}_2(\theta) = W(\theta; \hat{\xi}) + K - \hat{c}_1(\theta_1)$, where $K = \mathbb{E}^{\tilde{\theta}|\theta_1}[V(\tilde{\theta}); \langle \xi, c \rangle]$ is the lowest period-1's type expected payoff under the original policies $\langle \xi, c \rangle$. From the properties (a)-(c) of $\hat{\xi}_2$ in the previous lemma, it is easy to see that, for each type θ_1 , $\mathbb{E}^{\tilde{\theta}|\theta_1}[V(\tilde{\theta}); \langle \hat{\xi}, \hat{c} \rangle] = \mathbb{E}^{\tilde{\theta}|\theta_1}[V(\tilde{\theta}); \langle \xi, c \rangle]$, and that the policies $\langle \hat{\xi}, \hat{c} \rangle$ satisfy all the conditions in Proposition 1 and hence are implementable (in particular, note that if (ξ_1, ξ_2) satisfy all the integral monotonicity conditions, so do $(\hat{\xi}_1, \hat{\xi}_2)$). Now, recall that the principal's payoff when the manager is risk neutral is given by the expression in (9). It is then easy to see that, for any θ_1 for which $\xi_2(\theta_1, \cdot)$ fails to be non-increasing in θ_2 , the difference in expected profits under $\langle \hat{\xi}, \hat{c} \rangle$ relative to $\langle \xi, c \rangle$ is given by (27), which is strictly positive. To establish the result it then suffices to note that, for each θ_1 for which the original policy $\xi_2(\theta_1, \cdot)$ fails to be non-increasing in θ_2 , one can choose (δ^*, δ^{**}) , as a function of θ_1 , so as to guarantee that the new policy $\hat{\xi}_2$ remains integrable over $\Theta = \Theta_1 \times \Theta_2$. ■

Step 2. Given Step 1, assume without loss of optimality that $\xi_2^*(\theta_1, \cdot)$ is non-increasing, for all θ_1 . Next recall that incentive compatibility requires that $\pi_1(\theta_1) + \gamma \mathbb{E}^{\tilde{\theta}_2|\theta_1} [\pi_2(\theta_1, \tilde{\theta}_2)]$ be non-decreasing in θ_1 ; i.e. $\xi_1^*(\theta_1) + \theta_1 + \gamma \mathbb{E}^{\tilde{\theta}_2|\theta_1} [\xi_2^*(\theta_1, \tilde{\theta}_2) + \gamma \theta_1]$ must be non-decreasing. Furthermore, from (13), at the optimum, for almost all θ_1 , $\mathbb{E}^{\tilde{\theta}_2|\theta_1} [\xi_2^*(\theta_1, \tilde{\theta}_2)] = \gamma \xi_1^*(\theta_1) + 1 - \gamma$, where we used the fact that $\psi'(\xi) = \xi$. It follows that $\xi_1^*(\theta_1) + \theta_1$ must be non-decreasing. Now suppose that the claim in the proposition is not true. Using again the fact that $\psi'(\xi) = \xi$, we then have that, for any $\eta > 0$, there is a positive-measure set of θ_1 such that

$$\xi_1^*(\theta_1) + \gamma \mathbb{E}^{\tilde{\theta}_2|\theta_1} [\xi_2^*(\theta_1, \tilde{\theta}_2)] = \xi_1^*(\theta_1) + \gamma [\gamma \xi_1^*(\theta_1) + 1 - \gamma] < \eta, \quad (28)$$

or, equivalently, $\xi_1^*(\theta_1) < [\eta - \gamma(1 - \gamma)] / (1 + \gamma^2)$. We use this observation to show the following.

Lemma 7 *Suppose that $\sup \{[1 - F_1(\theta_1)]/f_1(\theta_1)\} < (1 + \gamma)/(1 + \gamma^2) - (\bar{\theta}_1 - \underline{\theta}_1)$ and that $F_2(\cdot|\cdot)$ satisfies the monotone-likelihood-ratio property. Let*

$$L_1 \equiv -\frac{\gamma(1 - \gamma)}{1 + \gamma^2} + \frac{1}{4} \left(1 - (\bar{\theta}_1 - \underline{\theta}_1) + \frac{\gamma(1 - \gamma)}{1 + \gamma^2} - \sup \left\{ \frac{1 - F_1(\theta_1)}{f_1(\theta_1)} \right\} \right)$$

and

$$L_2 \equiv 1 - \sup \left\{ \frac{1 - F_1(\theta_1)}{f_1(\theta_1)} \right\} - \frac{1}{4} \left(1 - (\bar{\theta}_1 - \underline{\theta}_1) + \frac{\gamma(1 - \gamma)}{1 + \gamma^2} - \sup \left\{ \frac{1 - F_1(\theta_1)}{f_1(\theta_1)} \right\} \right).$$

Suppose that, for any $\eta > 0$, there exists a positive-measure set $\hat{\Theta}_1(\eta) \subset \Theta_1$ such that $\xi_1^*(\theta_1) + \gamma \mathbb{E}^{\tilde{\theta}_2|\theta_1} [\xi_2^*(\theta_1, \tilde{\theta}_2)] < \eta$ for all $\theta_1 \in \hat{\Theta}_1(\eta)$. Then, there exists $\theta_1^\# \in [\underline{\theta}_1, \bar{\theta}_1]$ and $e^\# \in [L_1, L_2]$ such that, for all $\theta_1 < \theta_1^\#$, $\xi_1^*(\theta_1) \leq e^\#$, while for all $\theta_1 > \theta_1^\#$, $\xi_1^*(\theta_1) \geq e^\#$.

Proof. First note that the assumptions in the lemma imply that there exists a positive-measure set $\Theta_1' \subset \Theta_1$ such that $\xi_1^*(\theta_1) < L_1$ for all $\theta_1 \in \Theta_1'$. To see this, let

$$\eta = \frac{1 + \gamma^2}{4} \left(1 - (\bar{\theta}_1 - \underline{\theta}_1) + \frac{\gamma(1 - \gamma)}{1 + \gamma^2} - \sup \frac{1 - F_1(\theta_1)}{f_1(\theta_1)} \right)$$

in (28) and note that $\eta > 0$ under the assumptions in the lemma. Next observe that, for ξ^* to be optimal, there must exist a positive-measure set $\Theta_1'' \subset \Theta_1$ such that $\xi_1^*(\theta_1) \geq \xi_1^R(\theta_1) = 1 - \frac{1 - F_1(\theta_1)}{f_1(\theta_1)} > L_2$ for all $\theta_1 \in \Theta_1''$. If this was not the case, the principal could increase her payoff by increasing $\xi_1^*(\theta_1)$ uniformly across Θ_1 by $\varepsilon > 0$, leaving $c_1(\cdot)$ and $\xi_2(\cdot)$ unchanged, and then adjusting the period-2 compensation c_2 so as to satisfy (3) while continuing to give the lowest period-1 type the same payoff $K = \mathbb{E}^{\tilde{\theta}_1|\theta_1}[V(\tilde{\theta}); \langle \xi^*, c^* \rangle]$ as under the original policies $\langle \xi^*, c^* \rangle$. This would relax the participation constraints (use (6) to see it), would not affect integral monotonicity, and would bring the period-1 policy closer to the one ξ_1^R that maximizes virtual surplus, thus improving the principal's expected payoff, as given by (9).

In what follows, we show that, since $\xi_1^*(\theta_1) + \theta_1$ is non-decreasing, there exists $\theta_1^\# \in [\underline{\theta}_1, \bar{\theta}_1]$ and $e^\# \in [L_1, L_2]$ such that, for all $\theta_1 < \theta_1^\#$, $\xi_1^*(\theta_1) \leq e^\#$, while for all $\theta_1 > \theta_1^\#$, $\xi_1^*(\theta_1) \geq e^\#$. We establish the result by contradiction. Suppose the claim in the lemma is not true. This means that the following result must instead be true:

Claim A: For all $e \in [L_1, L_2]$, all $\theta_1^\# \in [\underline{\theta}_1, \bar{\theta}_1]$, there exists $\theta_1 < \theta_1^\#$ such that $\xi_1^*(\theta_1) > e$, or $\theta_1 > \theta_1^\#$ such that $\xi_1^*(\theta_1) < e$.

Now suppose Claim A is true. Let $[\cdot]^- : \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by $[a]^- = \max\{-a, 0\}$. Our goal is to construct a partition $\{y_0, y_1, \dots, y_m\}$, $m \in \mathbb{N}$, $\underline{\theta}_1 = y_0 < y_1 < \dots < y_{m-1} < y_m = \bar{\theta}_1$, of Θ_1 such that $\sum_{k=0}^{m-1} [\xi_1^*(y_{k+1}) - \xi_1^*(y_k)]^- > \bar{\theta}_1 - \underline{\theta}_1$, establishing that the negative variation of ξ_1^* over Θ_1 , i.e., the supremum of $\sum_{k=0}^{m-1} [\xi_1^*(y_{k+1}) - \xi_1^*(y_k)]^-$ over all partitions of Θ_1 , exceeds $\bar{\theta}_1 - \underline{\theta}_1$. We know this to be incompatible with the fact that $\xi_1^*(\theta_1) + \theta_1$ is non-decreasing over Θ_1 , establishing that Claim A must be false.

For any $e \in [L_1, L_2]$, let $\theta_e^\# = \inf \left\{ \theta_1 : \xi_1^*(\tilde{\theta}_1) \geq e \text{ for all } \tilde{\theta}_1 > \theta_1 \right\}$. By the definition of $\theta_e^\#$, for all $\epsilon > 0$, there must exist $\theta_1 \in (\theta_e^\# - \epsilon, \theta_e^\#)$ such that $\xi_1^*(\theta_1) < e$. Furthermore, again by definition of $\theta_e^\#$, for all $\theta_1 > \theta_e^\#$, $\xi_1^*(\theta_1) \geq e$. Hence, for Claim A to hold, there must exist $\theta_1', \theta_1'' < \theta_e^\#$, $\theta_1' < \theta_1''$, such that $\xi_1^*(\theta_1') > e > \xi_1^*(\theta_1'')$. Now, for each $e \in [L_1, L_2]$, let

$$b_e \equiv \sup \left\{ \xi_1^*(\theta_1) : \theta_1 < \theta_1' \text{ for some } \theta_1' \text{ for which } \xi_1^*(\theta_1') < e \right\} = \sup \left\{ \xi_1^*(\theta_1) : \theta_1 < \theta_e^\# \right\}$$

and

$$l_e \equiv \inf \left\{ \xi_1^*(\theta_1) : \text{for all } \epsilon > 0, \theta_1 > \theta_1' \text{ for some } \theta_1' \text{ with } \xi_1^*(\theta_1') > b_e - \epsilon \right\}.$$

Note that $C = \{(l_e, b_e) : e \in [L_1, L_2]\}$ is an open cover for $[L_1, L_2]$. To see this, note that, for each $e \in [L_1, L_2]$, $l_e < e < b_e$. By the Lindelof property of the real line, there exists a countable sub-cover $D = \{(l_{e_i}, b_{e_i}) : i \in \mathbb{N}\}$ of C , where $(e_i)_{i=1}^\infty$ is a sequence of points in $[L_1, L_2]$. Now let $\lambda(\cdot)$ be the Lebesgue measure. Then $\lambda(\cup_{i=1}^\infty (l_{e_i}, b_{e_i})) \geq L_2 - L_1$ and, for any $\epsilon > 0$, there exists n such that $\lambda(\cup_{i=1}^n (l_{e_i}, b_{e_i})) > L_2 - L_1 - \epsilon$. The following must then also be true.

Property A. *Suppose that Claim A is true. Then for any $n \in \mathbb{N}$, any $\epsilon > 0$, there exists a partition $\{y_0, y_1, \dots, y_m\}$, $m \in \mathbb{N}$, $\underline{\theta}_1 = y_0 < y_1 < \dots < y_{m-1} < y_m = \bar{\theta}_1$, of Θ_1 such that*

$$\sum_{k=0}^{m-1} [\xi_1^*(y_{k+1}) - \xi_1^*(y_k)]^- \geq \lambda(\cup_{i=1}^n (l_{e_i}, b_{e_i})) - \epsilon. \quad (29)$$

Proof of Property A. Fix $n \in \mathbb{N}$ and $\epsilon > 0$. Note that there is no loss in assuming that the cover D comprises only distinct sets; i.e., $b_{e_i} \neq b_{e_{i'}}$ for all $i \neq i'$. Since n is finite, we can take the values of e_i to be ordered: i.e., $e_1 < \dots < e_{n-1} < e_n$. Let $y_0 = \underline{\theta}_1$. Choose y_1 such that $y_1 < \theta_{e_1}^\#$ and $\xi_1^*(y_1) > b_{e_1} - \epsilon/2n$, together with $y_2 \in (y_1, \theta_{e_1}^\#)$ such that $\xi_1^*(y_2) < l_{e_1} + \epsilon/2n$. It should be clear from the definitions of l_e and b_e that these choices are possible. If the partition has been determined up to y_{2k} , then take $y_{2k+1} \in [\theta_{e_k}^\#, \theta_{e_{k+1}}^\#)$ such that $\xi_1^*(y_{2k+1}) > b_{e_{k+1}} - \epsilon/2n$ and $y_{2k+2} \in (y_{2k+1}, \theta_{e_{k+1}}^\#)$ such that $\xi_1^*(y_{2k+2}) < l_{e_{k+1}} + \epsilon/2n$. Proceeding this way, the partition is determined up to y_{2n} , and we then let $y_{2n+1} = \bar{\theta}_1$ (so that $m = 2n + 1$). Then it is easy to see that

$$\sum_{k=0}^{m-1} [\xi_1^*(y_{k+1}) - \xi_1^*(y_k)]^- \geq \sum_{i=1}^n \left(b_{e_i} - l_{e_i} - \frac{\epsilon}{n} \right) = \sum_{i=1}^n (b_{e_i} - l_{e_i}) - \epsilon \geq \lambda(\cup_{i=1}^n (l_{e_i}, b_{e_i})) - \epsilon.$$

This establishes Property A.

We therefore conclude that, for any $\epsilon > 0$, there exists a partition $\{y_0, y_1, \dots, y_m\}$, $m \in \mathbb{N}$, $\underline{\theta}_1 = y_0 < y_1 < \dots < y_{m-1} < y_m = \bar{\theta}_1$, of Θ_1 such that $\sum_{k=0}^{m-1} [\xi_1^*(y_{k+1}) - \xi_1^*(y_k)]^- > L_2 - L_1 - 2\epsilon$. Because $L_2 - L_1 > \bar{\theta}_1 - \underline{\theta}_1$, there then exists a partition such that $\sum_{k=0}^{m-1} [\xi_1^*(y_{k+1}) - \xi_1^*(y_k)]^- > \bar{\theta}_1 - \underline{\theta}_1$. This shows that the negative variation of ξ_1^* over Θ_1 must be strictly larger than $\bar{\theta}_1 - \underline{\theta}_1$, as desired. ■

Now suppose that, for any $\eta > 0$, there exists a positive-measure set $\hat{\Theta}_1(\eta) \subset \Theta_1$ such that $\xi_1^*(\theta_1) + \gamma \mathbb{E}^{\tilde{\theta}_2 | \theta_1} [\xi_2^*(\theta_1, \tilde{\theta}_2)] < \eta$ for all $\theta_1 \in \hat{\Theta}_1(\eta)$. The result in the previous lemma implies that there exists $\theta_1^\# \in [\underline{\theta}_1, \bar{\theta}_1]$ and $e^\# \in [L_1, L_2]$ such that, for all $\theta_1 < \theta_1^\#$, $\xi_1^*(\theta_1) \leq e^\#$, while for all $\theta_1 > \theta_1^\#$, $\xi_1^*(\theta_1) \geq e^\#$. It is also easy to see that $\theta_1^\# > \underline{\theta}_1$, and that $\xi_1^*(\theta_1) < e^\#$ for a positive-measure subset of $[\underline{\theta}_1, \theta_1^\#]$ (both properties follow from the fact that, if they were not true, then $\xi_1^*(\theta_1) + \gamma \mathbb{E}^{\tilde{\theta}_2 | \theta_1} [\xi_2^*(\theta_1, \tilde{\theta}_2)]$ would be bounded away from zero from below with probability one, along with the fact that $\mathbb{E}^{\tilde{\theta}_2 | \theta_1} [\xi_2^*(\theta_1, \tilde{\theta}_2)] = \gamma \xi_1^*(\theta_1) + 1 - \gamma$). Then consider the alternative effort policy $\hat{\xi}$ defined by

$$\hat{\xi}_1(\theta_1) = \begin{cases} \xi_1^*(\theta_1) & \text{if } \theta_1 > \theta_1^\# \\ e^\# & \text{if } \theta_1 \leq \theta_1^\# \end{cases} \quad \text{and} \quad \hat{\xi}_2(\theta_1, \theta_2) = \begin{cases} \xi_2^*(\theta_1, \theta_2) & \text{if } \theta_1 > \theta_1^\# \\ 1 - \gamma + \gamma e^\# & \text{if } \theta_1 \leq \theta_1^\# \end{cases}$$

along with the compensation policy \hat{c} defined by $\hat{c}_1(\theta_1) = c_1^*(\theta_1)$ all θ_1 , and $\hat{c}_2(\theta_1, \theta_2) = W(\theta; \hat{\xi}) + K - \hat{c}_1(\theta_1)$, where $K = \mathbb{E}^{\tilde{\theta}_1} \left[V(\tilde{\theta}); \langle \xi^*, c^* \rangle \right]$ is the lowest period-1 type's expected payoff under the original policies $\langle \xi^*, c^* \rangle$. Now recall that the principal's payoff under any pair of implementable policies is given by (9). Further notice that the expression in (9) is strictly concave in the policies ξ and recall that (9) reaches its maximum at the policy ξ^R given by (18) and (19). Now note that, for all $\theta_1 \leq \theta_1^\#$, $\xi_1^*(\theta_1) \leq \hat{\xi}_1(\theta_1) \leq \xi_1^R(\theta_1)$, with the first inequality strict over a positive measure set of θ_1 (That $\hat{\xi}_1(\theta_1) \leq \xi_1^R(\theta_1)$ follows from the fact that $\hat{\xi}_1(\theta_1) = e^\# \leq L_2$, along with the fact, by definition of L_2 and of $\xi_1^R(\theta_1)$, $L_2 < \xi_1^R(\theta_1)$). Also, for all $\theta_1 \leq \theta_1^\#$, all θ_2 ,

$$\begin{aligned} \mathbb{E}^{\tilde{\theta}_2|\theta_1} \left[\xi_2^*(\theta_1, \tilde{\theta}_2) \right] &= \gamma \xi_1^*(\theta_1) + 1 - \gamma \leq \hat{\xi}_2(\theta_1, \theta_2) = \gamma \hat{\xi}_1(\theta_1) + 1 - \gamma \\ &\leq \xi_2^R(\theta_1, \theta_2) = \gamma \xi_1^R(\theta_1) + 1 - \gamma, \end{aligned}$$

where, again, the first inequality is strict over a positive measure set of θ_1 . For all $\theta_1 > \theta_1^\#$, instead, $\hat{\xi}_1(\theta_1) = \xi_1^*(\theta_1)$ and $\hat{\xi}_2(\theta_1, \cdot) = \xi_2^*(\theta_1, \cdot)$. It is then clear that, if the policies $\langle \hat{\xi}, \hat{c} \rangle$ are implementable, they lead to higher expected profits than the policies $\langle \xi^*, c^* \rangle$. In what follows we show that indeed, they are implementable. To see this, note that, for all θ_1 ,

$$\mathbb{E}^{\tilde{\theta}_1} \left[V(\tilde{\theta}); \langle \hat{\xi}, \hat{c} \rangle \right] \geq \mathbb{E}^{\tilde{\theta}_1} \left[V(\tilde{\theta}); \langle \xi^*, c^* \rangle \right]$$

which implies that $\langle \hat{\xi}, \hat{c} \rangle$ satisfy all the participation constraints. Next observe that, for all θ_1 , $\pi_2(\theta_1, \cdot)$ is non-decreasing in θ_2 and that,

$$\pi_1(\theta_1) + \gamma \mathbb{E}^{\tilde{\theta}_2|\theta_1} \left[\pi_2(\theta_1, \tilde{\theta}_2) \right] = \theta_1 + \hat{\xi}_1(\theta_1) + \gamma \left\{ 1 - \gamma + \gamma \hat{\xi}_1(\theta_1) + \mathbb{E}^{\tilde{\theta}_2|\theta_1} \left[\tilde{\theta}_2 \right] \right\}$$

is non-decreasing in θ_1 (these properties follow directly from the way $\hat{\xi}$ is constructed along with the fact that, to be optimal, the policy ξ^* must satisfy the condition $\mathbb{E}^{\tilde{\theta}_2|\theta_1} \left[\xi_2^*(\theta_1, \tilde{\theta}_2) \right] = \gamma \xi_1^*(\theta_1) + 1 - \gamma$). Next, observe that, by construction, the compensation policy \hat{c} satisfies Condition (3). It thus suffices to show that the new effort policy $\hat{\xi}$ satisfies the integral monotonicity constraints of Proposition 1. That is, for all $\theta_1, \hat{\theta}_1 \in \Theta_1$,

$$\int_{\hat{\theta}_1}^{\theta_1} \left\{ \hat{\xi}_1(\hat{\theta}_1) - s + \hat{\theta}_1 + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[\hat{\xi}_2(\hat{\theta}_1, \tilde{\theta}_2) \right] \right\} ds \leq \int_{\hat{\theta}_1}^{\theta_1} \left\{ \hat{\xi}_1(s) + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[\hat{\xi}_2(s, \tilde{\theta}_2) \right] \right\} ds.$$

The only two cases which are not immediate are (i) $\hat{\theta}_1 \leq \theta_1^\# < \theta_1$, and (ii) $\theta_1 \leq \theta_1^\# < \hat{\theta}_1$. For Case (i), because $\hat{\xi}_1(\cdot)$ and $\hat{\xi}_2(\cdot)$ are constant over any (θ_1, θ_2) such that $\theta_1 \leq \theta_1^\#$, it is enough to show that, for any $s > \theta_1^\#$,

$$\hat{\xi}_1(s) + s + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[\hat{\xi}_2(s, \tilde{\theta}_2) \right] \geq \hat{\xi}_1(\hat{\theta}_1) + \hat{\theta}_1 + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[\hat{\xi}_2(\hat{\theta}_1, \tilde{\theta}_2) \right].$$

This follows from the fact that

$$\begin{aligned} \hat{\xi}_1(s) + s + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[\hat{\xi}_2(s, \tilde{\theta}_2) \right] &= \hat{\xi}_1(s) + s + \gamma \left(\gamma \hat{\xi}_1(s) + 1 - \gamma \right) \geq \hat{\xi}_1(\theta_1^\#) + \theta_1^\# + \gamma \left(\gamma \hat{\xi}_1(\theta_1^\#) + 1 - \gamma \right) \\ &\geq \hat{\xi}_1(\hat{\theta}_1) + \hat{\theta}_1 + \gamma \left(\gamma \hat{\xi}_1(\hat{\theta}_1) + 1 - \gamma \right) = \hat{\xi}_1(\hat{\theta}_1) + \hat{\theta}_1 + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[\hat{\xi}_2(\hat{\theta}_1, \tilde{\theta}_2) \right], \end{aligned}$$

where the inequalities follow from the fact that the original policy is such that $\xi_1^*(s) + s$ is non-decreasing, with $\xi_1^*(s) \geq e^\# = \hat{\xi}_1(\hat{\theta}_1)$ for all $s > \theta_1^\#$.

For Case (ii), note first that integral monotonicity requires that

$$\int_{\theta_1}^{\hat{\theta}_1} \left\{ \hat{\xi}_1(\hat{\theta}_1) - s + \hat{\theta}_1 + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[\hat{\xi}_2(\hat{\theta}_1, \tilde{\theta}_2) \right] \right\} ds \geq \int_{\theta_1}^{\hat{\theta}_1} \left\{ \hat{\xi}_1(s) + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[\hat{\xi}_2(s, \tilde{\theta}_2) \right] \right\} ds.$$

Because the original policy satisfies integral monotonicity, and because $(\hat{\xi}_1(\hat{\theta}_1), \hat{\xi}_2(\hat{\theta}_1, \tilde{\theta}_2))$ coincides with the original policy $(\xi_1^*(\tilde{\theta}_1), \xi_2^*(\tilde{\theta}_1, \tilde{\theta}_2))$ for any $(\tilde{\theta}_1, \tilde{\theta}_2)$ such that $\tilde{\theta}_1 > \theta_1^\#$, it suffices to show that

$$\int_{\theta_1}^{\theta_1^\#} \left\{ \hat{\xi}_1(\hat{\theta}_1) - s + \hat{\theta}_1 + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[\hat{\xi}_2(\hat{\theta}_1, \tilde{\theta}_2) \right] \right\} ds \geq \int_{\theta_1}^{\theta_1^\#} \left\{ \hat{\xi}_1(s) + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[\hat{\xi}_2(s, \tilde{\theta}_2) \right] \right\} ds.$$

To see this, it suffices to show that, for any $s < \theta_1^\#$

$$\hat{\xi}_1(s) + s + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[\hat{\xi}_2(s, \tilde{\theta}_2) \right] - \left(\hat{\xi}_1(\hat{\theta}_1) + \hat{\theta}_1 + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[\hat{\xi}_2(\hat{\theta}_1, \tilde{\theta}_2) \right] \right) \leq 0. \quad (30)$$

To prove that this is the case, first note that, for all $s < \theta_1^\#$, all $\theta'_1 \geq \theta_1^\#$,

$$\begin{aligned} & \hat{\xi}_1(s) + s + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[\hat{\xi}_2(s, \tilde{\theta}_2) \right] - \left\{ \hat{\xi}_1(\hat{\theta}_1) + \hat{\theta}_1 + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[\hat{\xi}_2(\hat{\theta}_1, \tilde{\theta}_2) \right] \right\} \\ &= e^\# + s + \gamma \left[1 - \gamma + \gamma e^\# \right] - \left\{ \hat{\xi}_1(\hat{\theta}_1) + \hat{\theta}_1 + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[\hat{\xi}_2(\hat{\theta}_1, \tilde{\theta}_2) \right] \right\} \\ &\leq \hat{\xi}_1(\theta'_1) + \theta'_1 + \gamma \left[1 - \gamma + \gamma \hat{\xi}_1(\theta'_1) \right] - \left\{ \hat{\xi}_1(\hat{\theta}_1) + \hat{\theta}_1 + \gamma \mathbb{E}^{\tilde{\theta}_2|\theta'_1} \left[\hat{\xi}_2(\hat{\theta}_1, \tilde{\theta}_2) \right] \right\} \end{aligned}$$

where the inequality follows from the fact that $e^\# + s + \gamma \left[1 - \gamma + \gamma e^\# \right] \leq \hat{\xi}_1(\theta'_1) + \theta'_1 + \gamma \left[1 - \gamma + \gamma \hat{\xi}_1(\theta'_1) \right]$ and from the fact that $\hat{\xi}_2(\hat{\theta}_1, \cdot)$ is non-increasing, which implies that $\mathbb{E}^{\tilde{\theta}_2|s} \left[\hat{\xi}_2(\hat{\theta}_1, \tilde{\theta}_2) \right] \geq \mathbb{E}^{\tilde{\theta}_2|\theta'_1} \left[\hat{\xi}_2(\hat{\theta}_1, \tilde{\theta}_2) \right]$.

Finally observe that, because $(\hat{\xi}_1(\cdot), \hat{\xi}_2(\cdot))$ coincides with the original policy $(\xi_1^*(\cdot), \xi_2^*(\cdot))$ for any (θ_1, θ_2) such that $\theta_1 > \theta_1^\#$, the fact that ξ^* satisfies integral monotonicity implies that there must exist a $\theta'_1 \in (\theta_1^\#, \hat{\theta}_1)$ such that

$$\begin{aligned} & \hat{\xi}_1(\theta'_1) + \theta'_1 + \gamma \left[1 - \gamma + \gamma \hat{\xi}_1(\theta'_1) \right] - \left\{ \hat{\xi}_1(\hat{\theta}_1) + \hat{\theta}_1 + \gamma \mathbb{E}^{\tilde{\theta}_2|\theta'_1} \left[\hat{\xi}_2(\hat{\theta}_1, \tilde{\theta}_2) \right] \right\} \\ &= \hat{\xi}_1(\theta'_1) + \theta'_1 + \gamma \mathbb{E}^{\tilde{\theta}_2|\theta'_1} \left[\hat{\xi}_2(\theta'_1, \tilde{\theta}_2) \right] - \left\{ \hat{\xi}_1(\hat{\theta}_1) + \hat{\theta}_1 + \gamma \mathbb{E}^{\tilde{\theta}_2|\theta'_1} \left[\hat{\xi}_2(\hat{\theta}_1, \tilde{\theta}_2) \right] \right\} \leq 0. \end{aligned}$$

We conclude that, for all $s < \theta_1^\#$, the inequality in (30) holds. This completes the proof of the proposition. Q.E.D.

Proof of Proposition 6. See supplementary material.

Proof of Proposition 7. See supplementary material.

Proof of Proposition 8. To establish the necessity of (21) and (22), consider the perturbed effort policy $\xi_1(\theta_1) = \xi_1^R(\theta_1) + a\nu(\theta_1)$ and $\xi_2(\theta) = \xi_2^R(\theta) + b\omega(\theta)$ for scalars a and b and measurable

functions $\nu(\cdot)$ and $\omega(\cdot)$. Then differentiate the firm's profits (8) with respect to a and b respectively. A necessary condition for the proposed policy ξ^R to maximize (8) is that these derivatives, evaluated at $a = b = 0$ vanish for all measurable functions $\nu(\cdot)$ and $\omega(\cdot)$. This is true only if ξ^R satisfies (21) and (22) with probability one.

Uniqueness of ξ^R and c^R , as well as the necessity of (12), follow from the same arguments as in the proof of Proposition 2.