Dynamic mechanism design: dynamic arrivals and changing values*

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Abstract

We study the optimal mechanism in a dynamic sales relationship where the buyer's arrival date is uncertain, and where his value changes stochastically over time. The buyer's arrival date is the first date at which contracting is feasible and is his private information. To induce immediate participation, the buyer is granted positive expected rents even if his value at arrival is the lowest possible. The buyer is punished for arriving late; i.e., he expects to earn less of the surplus. Optimal allocations for a late arriver are also further distorted below first-best levels. Conditions are provided under which allocations converge to the efficient ones long enough after contracting, and this convergence occurs irrespective of the time the contract is initially agreed (put differently, the so-called "principle of vanishing distortions" introduced by Battaglini (2005) continues to apply irrespective of the buyer's arrival date).

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*This paper supersedes the second half of my earlier work "Durable Goods Sales with Dynamic Arrivals and Changing Values", incorporating the ideas and results therein. The main difference with respect to that work is that I focus here on a time-separable problem, with the advantage that it simplifies some of the analysis and allows a greater focus on the key novelties of the problem. The paper has benefited from detailed comments from my PhD advisor, Alessandro Pavan, as well as the helpful suggestions of Simon Board, Rahul Deb, Jeff Ely, Igal Hendel, Konrad Mierendorff, Bill Rogerson, Bruno Strulovici and Rakesh Vohra. I am grateful for seminar participants at the various universities where this paper, and its earlier incarnations, were presented.

1 Introduction

Markets for most goods are highly dynamic. Buyers may become interested in acquiring goods at different times, such as when they first encounter advertisements for the product. Once in the market, their preferences can be expected to change. Buyers' eagerness to consume often hinges on their own circumstances. Purchasers of cellular telephone plans or wireless internet packages, for instance, have preferences that fluctuate with their available leisure time and contact with friends. Commercial buyers' needs may change in long-term supply relationships. For instance, a restaurant's preferences for acquiring high-quality ingredients from a supplier may vary with changes in its menu, which may come at the whim of the chef.

To examine a market with these features, we consider a buyer who has vertical preferences over the quality levels that the seller can supply. The buyer's arrival date to the market (which is the first date he can contract with the seller) is uncertain and, having arrived, his preferences evolve stochastically with time. The key difficulty for designing the profit-maximizing mechanism in such a setting is that the buyer is strategic, and can "hide" his (privately known) arrival to the market. That is, he may participate in the mechanism only at the moment of his choice. In particular, the buyer may prefer to wait to learn if his preferences will change before participating.

The aforementioned difficulty has been largely ignored by the dynamic mechanism design literature (we provide details on some exceptions in Section 5 below). That literature, for the most part, follows two main strands. One strand considers profit-maximizing mechanisms for agents whose preferences evolve stochastically with time and who are available to participate at the date the principal fixes the mechanism (see, e.g., Baron and Besanko (1984), Besanko (1985), Courty and Li (2000), Battaglini (2005), Eso and Szentes (2007), and Pavan, Segal and Toikka (forthcoming)). The other considers dynamic mechanisms when agents arrive over time but preferences do not change (see, e.g., Conlisk, Gerstner and Sobel (1984), Board (2008), Gershkov and Moldovanu (2009), Said (2012), Pai and Vohra (2013) and Board and Skrzypacz (forthcoming)).[1] While these strands have mainly developed independently (see Bergemann and Said (2011) for a summary), combining features from both is a step towards realism and allows us to uncover new tradeoffs.

The key properties of an optimal mechanism are as follows. First, because the buyer may arrive at any moment, the mechanism optimally permits participation at each possible arrival date (inducing participation at the buyer's arrival date is optimal by the revelation principle, appropriately extended to the dynamic environment). This contrasts with the first strand of literature mentioned above, where there is a single participation date, and where an agent who does not participate at the specified date is excluded from the mechanism forever. The possibility to delay participation means that if the buyer arrives to the market with a low value, he can delay participation until his

[1] There is also a literature with dynamic arrivals but without commitment; examples include Conlisk, Gerstner and Sobel (1984) and Dilme and Li (2016).

value becomes high. For this reason, inducing immediate participation means leaving the buyer with positive rent even if he arrives to the market with the lowest possible value.

That the buyer has the ability to wait and participate at a later date means that he has an *endogenous outside option*. The value of this option depends on how the mechanism treats later arrival. An optimal mechanism therefore punishes late arrival: If the buyer arrives late, then he faces worse terms of trade, purchases quality levels which are distorted further below their efficient levels, and expects to earn less rent. By lowering the option value of waiting, the seller extracts more of the surplus for herself. Our finding thus contrasts with the much simpler case of constant values, where the optimal mechanism involves a repetition of the static optimum, and where the buyer therefore receives the same treatment irrespective of the participation date. Because values are persistent in our setting, how the agent fares if delaying participation depends on his current value for the good, and this means the value of the agent's outside option is type dependent (see Jullien, 2000, for a study of (static) mechanism design with type-dependent outside options).

The quality levels supplied under a contract signed at a given date τ depend critically on the ratio between the probability of arrival at date τ and the probability of arrival at any earlier date. A smaller ratio implies that the seller cares relatively less about efficiency at date τ and more about limiting the rents available in case of arrival before τ. When the ratio decreases with time, the optimal quality allocations thus become increasingly (downward) distorted at later contracting dates. When the horizon is infinite, and when the buyer arrives at each date with positive probability, the ratio necessarily converges to zero with time. Under appropriate regularity conditions, the rents the buyer expects for an optimal mechanism then converge to zero as the participation date goes to infinity.

Although the buyer receives lower qualities if he arrives late, it is often still the case that quality prescriptions converge to their first-best levels after a sufficiently long relationship. Put differently, the "principle of vanishing distortions" first described by Battaglini (2005) and adapted to richer settings by Pavan, Segal and Toikka (2014) continues to hold. The reason is that quality choices at dates long after the relationship has commenced have little effect on the information rents that the buyer expects. This is familiar from the existing literature: loosely, the result is driven by the assumption that a buyer's initial value for quality is a poor predictor of his value far in the future.

Finally, note that, although we focus on a buyer-seller relationship, our approach is relevant for agency problems in other settings. A government which seeks to procure services at the least cost to tax payers may face new suppliers arising over time whose production costs can be expected to change. Firms seeking to fill top management positions may face potential managers who become available or learn of the position only after time, while their suitability for the job continues to change. Our focus on the seller's problem with vertical preferences over quality (as in Mussa and Rosen (1978)) is thus only for convenience, and because it allows us to draw comparisons to the existing literature, especially Battaglini (2005) and subsequent work (e.g., Boleslavsky and Said (2012) and Battaglini and Lamba (2015)). Our choice of setting simplifies the analysis and key insights relative to an earlier working paper version (Garrett, 2011). The earlier working paper differs from the present version in that it studies a durable-goods problem where buyers have unit demand, and have no preference for the good once this is satisfied. In contrast, the primitives in this paper are time separable in that buyer costs and seller preferences do not depend on past consumption.[2]

The rest of the paper is as follows. Section 2 introduces the model and do so for two specifications: two possible values and a continuum of values. Section 3 then analyzes the case with two values while Section 4 analyzes the case with a continuum. Section 5 provides further discussion of how the findings relate to other contributions in the literature. Section 6 concludes.

2 Model and Preliminaries

Basics. We consider a repeated buyer-seller relationship in discrete time, which lasts until the end of period $T \in \{2, \ldots, \infty\}$. The buyer values consumption of a non-durable good, which can be provided by the seller in each period. Both buyer and seller have a common discount factor $\delta \leq 1$ ($\delta < 1$ in case $T = \infty$).

The buyer arrives at some date $\tau \in \{1, \ldots, T\}$. This is the first date at which the buyer is available to communicate with the seller; contracting is impossible before this date. Inability to contract at earlier dates may reflect a range of reasons: the buyer may be entirely unaware of the seller's existence before encountering an advertisement which explains the mechanism, or he may be aware of the seller's offer but unable to communicate until the opportunity arises to physically meet.[3]

Payoffs. At each date after arrival, if the buyer purchases a good of quality q at date t, paying p, then he earns a payoff

$$\theta_{\tau,t} q - p$$

where $\theta_{\tau,t} \in \Theta$ is his value at date t, and where Θ is a bounded subset of $\mathbb{R}$.

The seller has a cost of producing q units equal to $c(q)$, where $c(\cdot)$ is a continuously-differentiable cost function defined on $[0, \bar{q}]$.[4] The cost function $c(\cdot)$ is strictly increasing, strictly convex, and

[2] Another difference is that we focus here on a discrete-time rather than continuous-time set-up. This makes the analysis at least conceptually simpler, permits the treatment of a continuum of types (as described below), and comes without loss of tractability. See a further discussion of the relation to the earlier (unpublished) paper in Section 5 below.

[3] Our notion of "arrival" is distinct from other notions that one might be tempted to use, such as the date a buyer first learns his value for the good. See Akan, Ata and Dana (2015) for a model where a buyer learns his value at different dates.

[4] We introduce the bound on quality for simplicity, as it guarantees the applicability of the dynamic envelope formula in Pavan, Segal and Toikka (2014) to our model with a continuum of values.

satisfies $c'(0) = c(0) = 0$ and $c'(\bar{q}) > \sup\Theta$.[5] The seller then earns a period$-t$ payoff from selling q units at price p equal to $p - c(q)$.

Distribution of buyer arrival dates. The probability of arrival at each date τ is $\rho_\tau \geq 0$, with $\sum_{\tau=1}^{T} \rho_\tau \leq 1$. For each τ, let $\beta_\tau = \sum_{s=1}^{\tau-1} \rho_s$ be the probability that the buyer arrives before date τ.

Process for values: We consider two classes of processes for the buyer's value. For the first, there are two possible values, while for the second, we consider a continuum. Both possibilities have been important in the dynamic mechanism design literature where agents preferences change with time, but where the initial contracting date is fixed (e.g., Battaglini, 2005, studies the case with two values, while Pavan, Segal and Toikka, 2014, study the case of a continuum of values). We adopt the following notation: if the buyer arrives at date τ, then a sequence of values from date t to $t' > t$, with $t \geq \tau$, is denoted $\theta_{\tau,t}^{t'} = \left(\theta_{\tau,t}, \ldots, \theta_{\tau,t'}\right)$.

The processes we consider satisfy the following restriction, which is particularly important for keeping the seller's problem tractable. The distribution of the buyer's value at each date after his arrival depends only on his value in the previous period, and neither on his earlier values nor on his arrival date. This implies that, at any date t, the period-t value $\theta_{\tau,t}$ is a sufficient statistic for later values.

Two values. In the case with two values, if the buyer arrives at date τ, then he draws a value $\theta_{\tau,\tau} \in \{\theta_L, \theta_H\}$ with $\theta_L < \theta_H$. The probability of drawing a high date-τ value is given by $\Pr\left(\tilde{\theta}_{\tau,\tau} = \theta_H\right) = \mu \in (0,1)$. Values at each date $t > \tau$ are determined by the transitions $\Pr\left(\tilde{\theta}_{\tau,t} = \theta_H | \tilde{\theta}_{\tau,t-1} = \theta_L\right) = \alpha_L \in (0,1)$ and $\Pr\left(\tilde{\theta}_{\tau,t} = \theta_H | \tilde{\theta}_{\tau,t-1} = \theta_H\right) = \alpha_H \in (0,1)$, with $\alpha_L < \alpha_H$. Thus, a high value at any date implies a greater likelihood of high values at future dates (put differently, the process satisfies the standard first-order stochastic dominance assumption).

Continuum of values. In the continuum-values case, in each period t after arrival, the buyer draws a value from $\Theta = \left[\underline{\theta}, \bar{\theta}\right]$. This is the smallest set to include all possible values. The buyer's initial type $\theta_{\tau,\tau}$ at arrival date τ is drawn from a continuously differentiable c.d.f. F_{In} with density f_{In} and support on Θ.

For each date $t > \tau$, if the buyer's date $t-1$ value is $\theta_{\tau,t-1} \in \Theta$, then his date-$t$ value is drawn according to a continuously differentiable c.d.f. $F_{Tr}(\cdot|\theta_{\tau,t-1})$ with density $f_{Tr}(\cdot|\theta_{\tau,t-1})$ and with support on $\left[\underline{\theta}_{Tr}(\theta_{\tau,t-1}), \bar{\theta}_{Tr}(\theta_{\tau,t-1})\right] \subset \Theta$. The function $F_{Tr}(\cdot|\cdot)$ is also continuously differentiable in its second argument. Following Garrett and Pavan (2012), we specify that, for any $\theta_{\tau,t-1}, \theta_{\tau,t} \in \Theta$,

$$-f_{Tr}(\theta_{\tau,t}|\theta_{\tau,t-1}) \leq \frac{\partial F_{Tr}(\theta_{\tau,t}|\theta_{\tau,t-1})}{\partial \theta_{\tau,t-1}} \leq 0.$$

[5] The latter assumption will guarantee that the solutions to the optimal quality schedules that we derive below remain strictly below $\bar{q}$.

The second inequality implies that the conditional distributions $F_{Tr}(\theta_{\tau,t}|\theta_{\tau,t-1})$ are ranked in terms of first-order stochastic dominance, while the first inequality ensures that we can apply a "dynamic revenue equivalence" result developed in Pavan, Segal and Toikka (2014).[6]

Mechanisms. Both the buyer's arrival time and the evolution of his value are his private information. The seller can fully commit to a dynamic mechanism. By the revelation principle, we restrict attention to incentive-compatible direct mechanisms. The buyer is asked to report his arrival date τ and initial value $\theta_{\tau,\tau}$, and then to report his subsequent values $\theta_{\tau,t}$ in each period $t > \tau$. If the buyer arrives at date τ, then he can report to the mechanism at any moment from that date onwards.

A mechanism $\Omega = \langle \mathbf{q}, \mathbf{p} \rangle$ is a collection of allocation rules $\mathbf{q} = \langle q_{\tau,t} \rangle_{1 \leq \tau \leq t}$ and payments $\mathbf{p} = \langle p_{\tau,t} \rangle_{1 \leq \tau \leq t}$. If the buyer reports to the mechanism at date τ, and then reports a sequence of values $\hat{\theta}^t_{\tau,\tau} = \left(\hat{\theta}_{\tau,\tau}, \ldots, \hat{\theta}_{\tau,t}\right) \in \Theta^{t-\tau}$, then he receives the quality $q_{\tau,t}\left(\hat{\theta}^t_{\tau,\tau}\right) \in [0, \bar{q}]$ and pays $p_{\tau,t}\left(\hat{\theta}^t_{\tau,\tau}\right) \in \mathbb{R}$ at date t. A buyer who reports to the mechanism at date τ is deemed to accept the offer and binds himself to participate at all future dates. As is the case elsewhere in the literature, given that we impose no cash constraints, our assumption that the buyer can fully commit comes at no loss of generality. Indeed, by appropriately structuring the timing of payments, the buyer can always be induced to continue participating at every subsequent date, irrespective of his realized values.

3 Two values

Consider the process defined above where the buyer has two possible values for the good. Fix a mechanism $\Omega = \langle \mathbf{q}, \mathbf{p} \rangle$ and consider a buyer who reports to the mechanism at date τ, makes reports $\hat{\theta}^{t-1}_{\tau,\tau}$ up to date $t-1$ (if any) and has a date$-t$ valuation $\theta_{\tau,t}$. The expected continuation payoff of this buyer if he plans to report truthfully at all future dates is

$$V^{\Omega}_{\tau,t}\left(\theta_{\tau,t}; \hat{\theta}^{t-1}_{\tau,\tau}\right) \equiv \mathbb{E}\left[\sum_{s=t}^{T} \delta^{s-t}\left(\theta_{\tau,s}\ q_{\tau,s}\left(\hat{\theta}^{t-1}_{\tau,\tau}, \tilde{\theta}^{s}_{\tau,t}\right) - p_{\tau,s}\left(\hat{\theta}^{t-1}_{\tau,\tau}, \tilde{\theta}^{s}_{\tau,t}\right)\right) \mid \tilde{\theta}_{\tau,t} = \theta_{\tau,t}\right].$$

Using the same arguments as in Battaglini (2005), we can establish the following useful result concerning how the buyer's continuation payoff at any date t depends on his date$-t$ value. To state it, we introduce the following notation: for any $k \in \{1, \ldots, T\}$, $\theta^k_L = (\theta_L, \ldots, \theta_L)$ is a sequence of low values of length k.

Lemma 1 (Battaglini, 2005) *Fix an incentive-compatible mechanism Ω, and consider a buyer who first reports at date τ, and then reports a sequence $\hat{\theta}^{t-1}_{\tau,\tau}$ up to date $t-1$ (or makes no reports*

[6] As noted in Garrett and Pavan (2012), the lower bound on $\frac{\partial F_{Tr}(\theta_t|\theta_{t-1})}{\partial \theta_{t-1}}$ is equivalent to the assumption that, for any $\theta_{t-1} \in \Theta$, and any $x \in \mathbb{R}$, $1 - F(\theta_{t-1} + x|\theta_{t-1})$ is nonincreasing in θ_{t-1}.

in case $t = \tau$). The buyer's expected payoff satisfies

$$V_{\tau,t}^{\Omega}\left(\theta_H;\hat{\theta}_{\tau,\tau}^{t-1}\right) - V_{\tau,t}^{\Omega}\left(\theta_L;\hat{\theta}_{\tau,\tau}^{t-1}\right) \geq (\theta_H - \theta_L)\sum_{s=t}^{T}\delta^{s-t}\left(\alpha_H - \alpha_L\right)^{s-t} q_{\tau,s}\left(\hat{\theta}_{\tau,\tau}^{t-1}, \theta_L^{s-t+1}\right). \tag{1}$$

Lemma 1 provides a lower bound on the additional payoff the buyer expects when his value is high rather than low at a given date t. One way to interpret the condition is as follows. First, suppose we adjust payments at dates $t+1$ onwards so that the payoffs satisfy (1) with equality at all such dates, and for all histories. Assuming the new mechanism is incentive compatible at dates $t+1$ onwards, the buyer is then willing to always report a low value at all such dates. We can then evaluate the buyer's incentive to misreport at date t under the assumption that he always reports a low value in future. A necessary condition for incentive compatibility at date t is then that the buyer expects a payoff from a high value at date t (i.e., $V_{\tau,t}^{\Omega}\left(\theta_H;\hat{\theta}_{\tau,\tau}^{t-1}\right)$) which exceeds that for a low value (i.e., $V_{\tau,t}^{\Omega}\left(\theta_L;\hat{\theta}_{\tau,\tau}^{t-1}\right)$) by more than the expected difference in valuations under a strategy of always reporting a low value. The right-hand side of (1) is this difference.

Next, we deduce a lower bound on $V_{\tau,\tau}^{\Omega}(\theta_L;\emptyset)$ (i.e., on the buyer's payoff participating at date τ with a low value) in an incentive-compatible and individually-rational mechanism. Here, we use the following requirement. A buyer who arrives at date τ with a value θ_L must prefer to participate at date τ rather than to delay participation until date $\tau+1$, reporting truthfully at all future dates. That is, for all dates τ,

$$V_{\tau,\tau}^{\Omega}(\theta_L;\emptyset) \geq \delta\left((1-\alpha_L)V_{\tau+1,\tau+1}^{\Omega}(\theta_L;\emptyset) + \alpha_L V_{\tau+1,\tau+1}^{\Omega}(\theta_H;\emptyset)\right). \tag{2}$$

This condition, together with the one given in Lemma 1, yields the following result.

Lemma 2 *Fix an incentive-compatible mechanism $\Omega = \langle \mathbf{q}, \mathbf{p}\rangle$. The expected payoff of a buyer who arrives at date τ with a low value must satisfy*

$$V_{\tau,\tau}^{\Omega}(\theta_L;\emptyset) \geq \alpha_L(\theta_H - \theta_L)\sum_{i=1}^{T-\tau}\sum_{s=\tau+i}^{T}\delta^{s-\tau}(\alpha_H - \alpha_L)^{s-\tau-i} q_{\tau+i,s}\left(\theta_L^{s-\tau-i+1}\right). \tag{3}$$

Lemma 2 provides a lower bound on payoffs that will turn out to be tight in the optimal mechanism (under a certain regularity condition to be specified momentarily). To begin understanding this expression, it is simplest to consider the case where T is finite. Since the buyer must be willing to participate if he arrives at date T with a low value, we have $V_{T,T}^{\Omega}(\theta_L;\emptyset) \geq 0$. Now consider the buyer who arrives at date $\tau = T-1$ with value θ_L. If the buyer chooses not to participate at $T-1$, then he will have the option to participate at date T with a high value with probability α_L. In this case, he earns a positive rent $V_{T,T}^{\Omega}(\theta_H;\emptyset)$, which is at least $(\theta_H - \theta_L)\,q_{T,T}(\theta_L)$ by (1). Hence, we have $V_{T-1,T-1}^{\Omega}(\theta_L;\emptyset) \geq \delta(\theta_H - \theta_L)\,q_{T,T}(\theta_L)$, which is (3) evaluated at $\tau = T-1$. We can then work recursively backwards to deduce lower bounds on the rents at earlier dates. For instance, to deduce a lower bound $V^{\Omega}_{T-2,T-2}(\theta_L;\emptyset)$, we observe that, if the buyer delays participation until the subsequent period, then he earns a rent $V^{\Omega}_{T-1,T-1}(\theta_L;\emptyset)$ (which is at least $\delta(\theta_H-\theta_L)\,q_{T,T}(\theta_L)$) in case his value remains low (with probability $1-\alpha_L$), or an additional rent $V^{\Omega}_{T-1,T-1}(\theta_H;\emptyset)-V^{\Omega}_{T-1,T-1}(\theta_L;\emptyset)$ satisfying (1) if his value turns high (with probability α_L).

Expression (3) is central to our analysis, for it shows how the rent that must be promised to ensure agent participation accumulates with time. When T is finite, the agent faces a standard outside option, normalized to zero. As just explained, ensuring participation at date $T-1$ requires ceding larger rents because the agent can wait for his value to increase at date T. In turn this raises the rents that must be promised to ensure participation at $T-2$.

In terms of characterizing the optimal mechanism, the value of Lemma 2 is that it allows us to find a convenient lower bound on buyer payoffs as a function of the quality allocations $\mathbf{q}$. In particular, Lemmas 1 and 2 together allow us to provide an upper bound on the achievable profit in an incentive-compatible mechanism, as stated in the next result. This bound coincides with the seller's profits in case all the inequalities in (1) and (3) hold as equalities. This upper bound, analogous to the "virtual surplus" in static mechanism design, turns out to be achievable under a mild condition on the arrival probability, which we describe below.

Lemma 3 *Suppose that Ω is an incentive-compatible, individually-rational mechanism implementing an allocation $\mathbf{q}$. Then expected profits are no greater than*

$$\mathbb{E}\left[\sum_{s=\tilde{\tau}}^{T}\delta^{s-1}\left(m^s_{\tilde{\tau}}\left(\tilde{\theta}^s_{\tilde{\tau},\tilde{\tau}}\right)q_{\tilde{\tau},s}\left(\tilde{\theta}^s_{\tilde{\tau},\tilde{\tau}}\right)-c\left(q_{\tilde{\tau},s}\left(\tilde{\theta}^s_{\tilde{\tau},\tilde{\tau}}\right)\right)\right)\right], \tag{4}$$

where, for all τ, and all $s\geq\tau$,

$$\begin{aligned} m^s_\tau\left(\theta_L^{s-\tau+1}\right) &= \theta_L-\left(\frac{\beta_\tau}{\rho_\tau}\frac{\alpha_L}{1-\mu}+\frac{\mu}{1-\mu}\right)\left(\frac{\alpha_H-\alpha_L}{1-\alpha_L}\right)^{s-\tau}(\theta_H-\theta_L)\,,\ \text{and} \\ m^s_\tau\left(\theta^s_{\tau,\tau}\right) &= \theta_{\tau,s}\ \text{for all}\ \theta^s_{\tau,\tau}\neq\theta_L^{s-\tau+1}, \end{aligned} \tag{5}$$

and where expectations are taken over the arrival time $\tilde{\tau}$, as well as the realized values $\tilde{\theta}^T_{\tilde{\tau},\tilde{\tau}}$.

It will be helpful to understand the virtual values m^s_τ in (5) as the surplus due to awarding additional quality at date s to a buyer who arrived at date τ, less the effect on the lowest feasible values of buyer rents. Condition (1) shows that the (lower bound on) additional rent a buyer expects when arriving with a high rather than a low value, i.e. $V^{\Omega}_{\tau,\tau}(\theta_H;\emptyset)-V^{\Omega}_{\tau,\tau}(\theta_L;\emptyset)$ for arrival date τ, depends only on the quality at histories where the buyer's value remains low. In turn, the bound on rents in (3) also depends only on the quality at these histories. Hence, at any history $\theta^s_{\tau,\tau}\neq\theta_L^{s-\tau+1}$, the virtual value corresponds simply to the buyer's value for quality $\theta_{\tau,s}$.

For any history $\theta^s_{\tau,\tau}$ where the buyer arrives at date τ and his value remains low until s, the virtual value of incremental quality is the buyer's value θ_L less a quantity that can be rewritten as

$$\frac{\beta_\tau\alpha_L(\alpha_H-\alpha_L)^{s-\tau}(\theta_H-\theta_L)+\rho_\tau\mu(\alpha_H-\alpha_L)^{s-\tau}(\theta_H-\theta_L)}{\rho_\tau(1-\mu)(1-\alpha_L)^{s-\tau}}. \tag{6}$$

Analogous to the distortion term in the virtual values of static mechanism design, this expression is the ratio of the effect of date-s quality $q_{\tau,s}\left(\theta_L^{s-\tau+1}\right)$ on buyer rents to the probability this quality is awarded. The second term in the numerator (i.e., $\rho_\tau\mu\left(\alpha_H-\alpha_L\right)^{s-\tau}\left(\theta_H-\theta_L\right)$) corresponds to the additional expected rents if the buyer happens to arrive at date τ with a high (rather than a low) value, an event which occurs with probability $\rho_\tau\mu$. The first term in the numerator (i.e., $\beta_\tau\alpha_L\left(\alpha_H-\alpha_L\right)^{s-\tau}\left(\theta_H-\theta_L\right)$) corresponds to the effect of increasing $q_{\tau,s}\left(\theta_L^{s-\tau+1}\right)$ on the rents earned in case of arrival at date $\tau-1$ or earlier (the probability of such an arrival time is β_τ, and how much rent the buyer expects at such dates depends on the rate at which a low value turns high, α_L, as explained in relation to Lemma 2). The denominator in (6) (i.e., $\rho_\tau\left(1-\mu\right)\left(1-\alpha_L\right)^{s-\tau}$) is simply the probability that the history $\theta_{\tau,\tau}^s=\theta_L^{s-\tau+1}$ occurs.

One can now choose the qualities which maximize the expression (4), and then verify that these qualities can be implemented as part of an incentive-compatible mechanism. This leads to the following result.

Proposition 1 *Suppose that, for all $\tau\leq T-1$, $m_{\tau+1}^{\tau+1}\left(\theta_L\right)\leq m_\tau^{\tau+1}\left(\theta_L^2\right)$. Profit-maximizing qualities $q_{\tau,s}^*$ are given, for each arrival date τ, each date $s\geq\tau$, and each $\theta_{\tau,\tau}^s$, by*

$$c'\left(q_{\tau,s}^*\left(\theta_{\tau,\tau}^s\right)\right)=\max\left\{m_\tau^s\left(\theta_{\tau,\tau}^s\right),0\right\}. \tag{7}$$

The proof proceeds by constructing a mechanism with allocations given by (7), such that all of the inequalities in (1) and (3) hold with equality. Buyer rents are then as small as possible for an incentive-compatible individually-rational mechanism implementing these allocations. Given that the allocations (7) maximize the expression in (4), the mechanism must maximize profits provided it is incentive compatible.

Optimal qualities balance the cost of providing a given quality level against the "virtual value" of provision introduced in Lemma 3. As discussed above, following sequences of low values, virtual values are less than the value to the buyer, capturing ex-ante expected buyer rents.

The allocation $q_{1,t}^*$ which applies when the buyer arrives at date $\tau=1$ is exactly the allocation that the seller would optimally choose in a problem where the buyer is *known* to arrive at date 1. Hence the allocation for a date-1 arrival is precisely the same as in Battaglini's (2005) paper, which did not study uncertain arrival times. This result is to be expected, since the allocation for date-1 arrival does not affect the rents that must be left in case of arrival after date 1. The only difference between the mechanism for $\tau=1$ in the present setting, and the one studied by Battaglini, lies in the prices paid (equivalently, the rent obtained) by the buyer. In the presenting setting, the buyer's payments must be lower so that the buyer is willing to participate at date 1 rather than delaying participation until later. As we have seen in relation to Lemma 2, inducing date-1 participation requires that the buyer expects a positive rent even if his initial value is low. In contrast, the buyer's outside option in Battaglini's analysis is equal to zero, since a buyer who does not participate at first instance is costlessly excluded by the seller at all future dates. A buyer with a low initial value in his model then expects zero rent under the optimal mechanism.

For arrival at dates $\tau > 1$, the optimal qualities at histories of low values, i.e. $q^*_{\tau,t}\left(\theta_L^{t-\tau+1}\right)$, are further distorted below first-best values. In particular, $q^*_{\tau,\tau+k}\left(\theta_L^{k+1}\right) \leq q^*_{1,1+k}\left(\theta_L^{k+1}\right)$ for all $\tau \geq 2$ and $k \in \{0, 1, \ldots, T-\tau\}$. These additional distortions reflect the seller's goal of reducing rents in case of arrival at a later date, in turn permitting a reduction of rents in case of earlier arrival (including possible arrival at date 1).

Note that Proposition 1 provides a sufficient condition for the incentive compatibility of our candidate mechanism in terms of the primitives of the problem. In particular, the assumption that $m_{\tau+1}^{\tau+1}(\theta_L) \leq m_\tau^{\tau+1}\left(\theta_L^2\right)$ is equivalent to

$$\frac{\beta_{\tau+1}}{\rho_{\tau+1}}\frac{\alpha_L}{1-\mu} + \frac{\mu}{1-\mu} \geq \frac{\alpha_H - \alpha_L}{1-\alpha_L}\left(\frac{\beta_\tau}{\rho_\tau}\frac{\alpha_L}{1-\mu} + \frac{\mu}{1-\mu}\right). \tag{8}$$

This condition guarantees that, for all $\tau \leq T-1$ and all $s \in \{\tau+1, \ldots, T\}$,

$$q^*_{\tau,s}\left(\theta_L^{s-\tau+1}\right) \geq q^*_{\tau+1,s}\left(\theta_L^{s-\tau}\right).$$

In other words, the assumption guarantees that a buyer receives a higher quality allocation if he participates in the mechanism one period earlier, even if his values turn out to remain low from the date of arrival. This ensures that, if the buyer's value is high, he expects a higher rent from immediate participation than by delaying (that this is true also when the buyer's value is low follows because the mechanism is constructed to satisfy (3); i.e., the buyer is precisely indifferent to participating and waiting one more period when his value is low).

From (8), it is easy to see that our sufficient condition is more likely to hold in case $\frac{\beta_\tau}{\rho_\tau}$ does not decrease too fast in τ or if the process is not too persistent, i.e. if $\frac{\alpha_H-\alpha_L}{1-\alpha_L}$ is small. It is always enough that $\frac{\beta_\tau}{\rho_\tau}$ is increasing in τ. It is therefore worth emphasizing that $\frac{\beta_\tau}{\rho_\tau}$ being increasing is a condition that holds for many natural distributions of arrival times. Since the probability of earlier arrival β_τ is increasing in τ, it suffices that ρ_τ is non-increasing; i.e., it is enough that earlier arrivals are more likely. For instance, if T is finite with $\rho_\tau = \frac{1}{T}$ for all τ, then $\frac{\beta_\tau}{\rho_\tau} = \tau - 1$. If arrivals are geometrically distributed with parameter $\lambda \in (0,1)$, i.e. $\rho_\tau = (1-\lambda)^{\tau-1}\lambda$ for all τ, then $\frac{\beta_\tau}{\rho_\tau} = \sum_{s=1}^{\tau-1}(1-\lambda)^{s-\tau}$.

If $\frac{\beta_\tau}{\rho_\tau}$ decreases too fast in τ, then profits equal to the the maximum of (4) may not be attainable in an incentive-compatible mechanism. In this case, one must resort to "ironing" to derive the optimal allocation. Roughly speaking, this requires raising the quality after histories of low values for earlier arrivals, and lowering them for later arrivals, as compared to the quality levels specified in (7). We do not study the ironed solution, but expect our key quatitative insights to carry over to settings where ironing is needed.

If $\frac{\beta_\tau}{\rho_\tau}$ is increasing in τ, then the weight the seller attributes to reducing the rent of earlier arrivers increases over time relative to the weight she assigns to the surplus generated in case of arrival at date

τ. Since buyer rents are determined by the qualities allocated in case the buyer's value remains low, this implies that these qualities are distorted downward more at later dates relative to the first-best levels. In particular, conditional on the buyer being in the relationship for the same length of time, distortions are greater if the buyer arrives later. Formally, we find the following.

Corollary 1 *Suppose that* $\frac{\beta_\tau}{\rho_\tau}$ *is increasing in* τ. *Consider any two dates* τ, τ', *with* $\tau < \tau'$, *and let* $k \in \{0, 1, \ldots, T - \tau'\}$. *Then*

$$q^*_{\tau,\tau+k}\left(\theta_L^{k+1}\right) \geq q^*_{\tau',\tau'+k}\left(\theta_L^{k+1}\right),$$

with a strict inequality in case $q^*_{\tau,\tau+k}\left(\theta_L^{k+1}\right) > 0$. *Moreover, if* $\theta_{\tau,\tau} = \theta_{\tau',\tau'}$, *then* $V^\Omega_{\tau,\tau}\left(\theta_{\tau,\tau};\emptyset\right) \geq V^\Omega_{\tau',\tau'}\left(\theta_{\tau',\tau'};\emptyset\right)$, *with a strict inequality if* $q^*_{\tau,s}\left(\theta_L^{s-\tau+1}\right) > 0$ *for some period* $s \geq \tau$.

The result indicates that, when $\frac{\beta_\tau}{\rho_\tau}$ is increasing with τ, late arrivers are punished in that they expect lower rents. This discourages delayed participation in the mechanism, allowing the seller to extract more rents from early arrivers. In particular, the quality choices are designed to discourage a buyer who arrives with a low value from delaying participation until his value becomes high. It therefore allows the seller to give up less rent in case of earlier arrival while inducing immediate participation.

When $T = +\infty$, and when the buyer arrives with positive probability at each date, the ratio $\frac{\beta_\tau}{\rho_\tau}$ necessarily approaches $+\infty$ with τ. As a consequence, we find that contracts become arbitrarily inefficient as the participation date grows large along histories where the buyer realizes only the low value θ_L.

Corollary 2 *Suppose that* $T = +\infty$, *with* $m^{\tau+1}_{\tau+1}\left(\theta_L\right) \leq m^{\tau+1}_{\tau}\left(\theta_L^2\right)$ *for all* τ *(a sufficient condition is that* $\frac{\beta_\tau}{\rho_\tau}$ *is increasing in* τ*). For any* s, *there exists* $\bar{\tau}$ *sufficiently large that* $q^*_{\tau,t}\left(\theta_L^{t-\tau+1}\right) = 0$ *for all* $\tau \geq \bar{\tau}$ *and all* t *such that* $t - \tau \leq s$. *Hence, buyer rents converge to zero with the participation date.*

What emerges then is a fairly robust principle that optimal mechanisms punish a late arriver. When the horizon is infinite, for instance, buyer rents become arbitrarily small with their participation date. Punishing very late arrivers is beneficial for the seller, since it permits a reduction in rents at all earlier dates, back to date 1.

In contrast, it is worth noting that the "vanishing distortions at the bottom" principle described by Battaglini (2005) continues to hold. In particular, we can show the following.

Corollary 3 *Suppose that* $T = +\infty$ *with* $m^{\tau+1}_{\tau+1}\left(\theta_L\right) \leq m^{\tau+1}_{\tau}\left(\theta_L^2\right)$ *for all* τ *(a sufficient condition is that* $\frac{\beta_\tau}{\rho_\tau}$ *is increasing in* τ*). For any arrival date* τ, $q^*_{\tau,t}\left(\theta_L^{t-\tau+1}\right)$ *converges to its efficient value* θ_L *as the length of the relationship* $t - \tau + 1$ *becomes large.*

The reason for this result is that quality choices at dates long after contracting have little effect on the buyer's rents; choosing qualities close to the efficient ones therefore costs the seller little in terms of the surplus that must be left to the buyer. This is easily seen from the inequality (1) in Lemma 1. According to this lemma, the additional rents that must be given to the buyer in case of arrival with a high value depends on the additional probability that the buyer has of a high value in future. Since $\alpha_H - \alpha_L < 1$, the additional probability of a high value vanishes with time, so later quality allocations affect the buyer's rents less (see Battaglini, 2005, for a more detailed explanation of the vanishing distortions property).

Finally, it is interesting to consider comparative statics on the transition probabilities. Virtual values and hence qualities for sequences of low values are decreasing in α_H. The reason is that, for higher α_H, a high value persists for a longer time, implying that qualities assigned for sequences of low values have a greater effect on the rents that must be left to the buyer in case his value is initially high (again, see the inequalities in (1) of Lemma 1). The parameter α_L, however, plays two roles. First, a higher value of α_L implies a smaller advantage of high values over low ones, i.e. the opposite effect as for α_H (see the inequalities in (1)). Second, it increases the likelihood that the buyer's value becomes high if his value is low and he delays participation in the mechanism. This in turn increases the option value of delaying participation. The seller's optimal response to the first effect is to increase qualities, while her optimal response to the second is the opposite. Whether qualities increase or decrease with α_L at any date then depends on parameters and the participation date.

3.1 Experience goods

One subtlety which we have so far overlooked is the possibility that the buyer's value evolves differently when consuming the good compared to when he is without it. For "experience goods", for instance, a buyer may learn about suitability through consumption, but otherwise learn only a little. More generally, the level of excitement a buyer has about a good (and the importance of high quality, in particular) might be expected to fluctuate differently depending on whether the buyer is consuming. In terms of our model, this means values switching at different rates before and after participating in the mechanism.

Let α_H^W and α_L^W, with $\alpha_L^W < \alpha_H^W$, denote the probabilities of a high value at date τ given, respectively, high and low values at date $\tau - 1$ when *not* consuming at $\tau - 1$. Maintain the existing notation for the probability of changes conditional on consumption (i.e., let α_H and α_L denote the probabilities of a high value at date t given high and low values when consuming at date $t-1$). The previous analysis is easily adapted to this setting, yielding the following result.

Proposition 2 *Suppose that $\frac{\beta_\tau}{\rho_\tau}$ is increasing in τ. Suppose that values evolve differently contingent on past consumption, as described above. Let virtual values be given, for all τ, all $s \geq \tau$, by*

$$\begin{aligned} m_\tau^{W,s}\left(\theta_L^{s-\tau+1}\right) &= \theta_L - \left(\frac{\beta_\tau}{\rho_\tau}\frac{\alpha_L^W}{1-\mu} + \frac{\mu}{1-\mu}\right)\left(\frac{\alpha_H - \alpha_L}{1-\alpha_L}\right)^{s-\tau}(\theta_H - \theta_L)\,, \text{ and} \\ m_\tau^{W,s}\left(\theta_{\tau,\tau}^s\right) &= \theta_{\tau,s} \text{ for all } \theta_{\tau,\tau}^s \neq \theta_L^{s-\tau+1}. \end{aligned}$$

Profit-maximizing qualities $q^W_{\tau,s}$ are given, for each arrival date τ, each date $s \geq \tau$, and each $\theta^s_{\tau,\tau}$, by

$$c'\left(q^W_{\tau,s}\left(\theta^s_{\tau,\tau}\right)\right) = \max\left\{m^{W,s}_{\tau}\left(\theta^s_{\tau,\tau}\right), 0\right\}. \quad (9)$$

A natural assumption is that $\alpha^W_L < \alpha_L$, with the implication that quality allocations are less distorted than those given in Proposition 1. In the extreme case where $\alpha^W_L = 0$, optimal allocations do not depend on the arrival date. The buyer earns no additional rents from his private information about arrival – i.e., a buyer who arrives with a low value expects zero rent irrespective of the arrival date. It is readily checked that the optimal qualities then coincide with those for the optimal mechanism with a known arrival date (as in Battaglini, 2005; equivalently, optimal qualities are the same as for date-1 arrival in Proposition 1). Otherwise, for $\alpha^W_L > 0$, our main qualitative predictions continue to hold; in particular, a buyer whose initial value is low expects positive rent, and (provided that $\frac{\beta_\tau}{\rho_\tau}$ is increasing in τ) quality allocations are less efficient the later the arrival date.

4 Continuum of values

We now turn to the case with a continuum of values. Following Pavan, Segal and Toikka (2014), we introduce the notion of "impulse responses". For any τ, t, s, with $\tau \leq t < s$, and any $\theta^s_{\tau,t}$, let

$$J^s_t\left(\theta^s_{\tau,t}\right) \equiv \Pi^s_{q=t+1}\left(-\frac{\partial F_{Tr}(\theta_{\tau,q}|\theta_{\tau,q-1})/\partial\theta_{\tau,q-1}}{f_{Tr}(\theta_{\tau,q}|\theta_{\tau,q-1})}\right).$$

If instead $s = t$, then $J^s_t\left(\theta^s_{\tau,t}\right) = J^t_t\left(\theta_{\tau,t}\right) = 1$. The value of the impulse response $J^s_t\left(\theta^s_{\tau,t}\right)$ can be interpreted as capturing the effect of an infinitesimal variation in $\theta_{\tau,t}$ on $\theta_{\tau,s}$. A simple example is any first-order autoregressive process with persistence parameter γ, in which case the impulse responses are independent of valuations and given by $J^s_t = \gamma^{s-t}$. A property that is helpful for understanding the impulse response function is that, given $\tau \leq t < s$,

$$\frac{\partial}{\partial\theta_{\tau,t}}\mathbb{E}\left[\tilde{\theta}_{\tau,s}|\tilde{\theta}_{\tau,t} = \theta_{\tau,t}\right] = \mathbb{E}\left[J^s_t\left(\tilde{\theta}^s_{\tau,t}\right)|\tilde{\theta}_{\tau,t} = \theta_{\tau,t}\right].$$

As for the two-type case, the buyer's expected payoff from truthful reporting in a mechanism $\Omega = \langle \mathbf{q}, \mathbf{p}\rangle$ is denoted $V^\Omega_{\tau,t}\left(\theta_{\tau,t}; \hat{\theta}^{t-1}_{\tau,\tau}\right)$. The following analog to Lemma 1 then determines how, if the mechanism is incentive compatible, the buyer's expected payoff must depend on his value at each date.

Lemma 4 (Pavan, Segal and Toikka, 2014) *Fix an incentive-compatible mechanism Ω, and consider a buyer who first reports at date τ, and then reports a sequence $\hat{\theta}^{t-1}_{\tau,\tau}$ up to date $t-1$ (or makes no reports in case $t = \tau$). If the buyer's value at date t is $\theta_{\tau,t}$, then his expected payoff satisfies*

$$V^\Omega_{\tau,t}\left(\theta_{\tau,t}; \hat{\theta}^{t-1}_{\tau,\tau}\right) = V^\Omega_{\tau,t}\left(\underline{\theta}; \hat{\theta}^{t-1}_{\tau,\tau}\right) + \int_{\underline{\theta}}^{\theta_{\tau,t}} \mathbb{E}\left[\sum_{s=t}^{T}\delta^{s-t}J^s_t\left(\tilde{\theta}^s_{\tau,t}\right)q_{\tau,s}\left(\hat{\theta}^{t-1}_{\tau,\tau}, \tilde{\theta}^s_{\tau,t}\right)|\tilde{\theta}_{\tau,t} = r\right]dr.$$

Our goal is to use this result to derive a lower bound on the buyer's expected payoff when arriving at any date τ, as in Lemma 2 for the two-type case. To do this we consider the incentive of a buyer who arrives at date τ with the lowest possible initial value $\underline{\theta}$ to misreport his arrival date by delaying participation until the following period. For this deviation to be unprofitable requires

$$V^{\Omega}_{\tau,\tau}\left(\underline{\theta};\emptyset\right) \geq \delta\mathbb{E}\left[V^{\Omega}_{\tau+1,\tau+1}\left(\tilde{\theta}_{\tau,\tau+1};\emptyset\right)|\tilde{\theta}_{\tau,\tau}=\underline{\theta}\right]. \tag{10}$$

Iterating this requirement yields the following result.

Lemma 5 *Fix an incentive-compatible mechanism $\Omega = \langle \mathbf{q}, \mathbf{p}\rangle$. The buyer's expected payoff for any arrival date τ must satisfy*

$$V^{\Omega}_{\tau,\tau}\left(\underline{\theta};\emptyset\right) \geq \sum_{i=1}^{T-\tau}\mathbb{E}\left[\frac{1-F_{Tr}\left(\tilde{\theta}_{\tau+i,\tau+i}|\underline{\theta}\right)}{f_{In}\left(\tilde{\theta}_{\tau+i,\tau+i}\right)}\sum_{t=\tau+i}^{T}\delta^{t-\tau}J^{t}_{\tau+i}\left(\tilde{\theta}^{t}_{\tau+i,\tau+i}\right)q_{\tau+i,t}\left(\tilde{\theta}^{t}_{\tau+i,\tau+i}\right)\right]. \tag{11}$$

Lemmas 4 and 5 can be used to provide the following analog of Lemma 3.

Lemma 6 *Suppose that Ω is an incentive-compatible, individually-rational mechanism implementing an allocation $\mathbf{q}$. Then expected profits are no greater than*

$$\mathbb{E}\left[\sum_{t=\tilde{\tau}}^{T}\delta^{s-1}\left(m^{t}_{\tilde{\tau}}\left(\tilde{\theta}^{t}_{\tilde{\tau},\tilde{\tau}}\right)q_{\tilde{\tau},t}\left(\tilde{\theta}^{t}_{\tilde{\tau},\tilde{\tau}}\right)-c\left(q_{\tilde{\tau},t}\left(\tilde{\theta}^{t}_{\tilde{\tau},\tilde{\tau}}\right)\right)\right)\right], \tag{12}$$

where, for all τ, all $t \geq \tau$, and all $\theta^{t}_{\tau,\tau}$,

$$m^{t}_{\tau}\left(\theta^{t}_{\tau,\tau}\right) = \theta_{\tau,t} - J^{t}_{\tau}\left(\theta^{t}_{\tau,\tau}\right)\left(\frac{\beta_{\tau}}{\rho_{\tau}}\frac{1-F_{Tr}\left(\theta_{\tau,\tau}|\underline{\theta}\right)}{f_{In}\left(\theta_{\tau,\tau}\right)}+\frac{1-F_{In}\left(\theta_{\tau,\tau}\right)}{f_{In}\left(\theta_{\tau,\tau}\right)}\right), \tag{13}$$

and where expectations are taken over the arrival time $\tilde{\tau}$, as well as the realized values $\tilde{\theta}^{T}_{\tilde{\tau},\tilde{\tau}}$.

We choose qualities to maximize the expression in (12). Under certain conditions, we can then find an incentive-compatible mechanism which implements these allocations, implying the following result.

Proposition 3 *Suppose that, (i) for all $\tau \leq T-1$, and all pairs $\theta^{\tau+1}_{\tau,\tau}$, $m^{\tau+1}_{\tau+1}\left(\theta_{\tau,\tau+1}\right) \leq m^{\tau+1}_{\tau}\left(\theta^{\tau+1}_{\tau,\tau}\right)$, and (ii) for all τ and all $t \geq \tau$, each $m^{t}_{\tau}(\cdot)$ is non-decreasing. Then profit-maximizing qualities are given, for all τ, all $t \geq \tau$, and all $\theta^{t}_{\tau,\tau}$, by*

$$c'\left(q^{*}_{\tau,t}\left(\theta^{t}_{\tau,\tau}\right)\right) = \max\left\{m^{t}_{\tau}\left(\theta^{t}_{\tau,\tau}\right),0\right\}.$$

Conditions (i) and (ii) of this proposition can be understood as follows. First, Condition (ii) guarantees the existence of a mechanism in which, once the buyer has accepted to participate, he truthfully reports his values at all dates. As discussed by Pavan, Segal and Toikka (2014), this condition can be relaxed, although the weaker conditions are often difficult to check. Condition (i) then plays the role of ensuring that the allocations which maximize (12) are implementable by an incentive-compatible mechanism in which the constraints in (11) are satisfied with equality.

Condition (i), which guarantees timely participation in the mechanism, is new relative to settings where the agent's arrival date is fixed or known (as in Pavan, Segal and Toikka, 2014, for instance). The mechanism we construct ensures that the inequality (11) holds as an *equality*, which means that the buyer is indifferent between participating and waiting to participate in the next period when his value is equal to $\underline{\theta}$. Condition (i) then implies that, under the allocations which maximize (12), immediate participation at date τ is preferred by all higher types. More precisely, it implies that the benefit of immediate participation is increasing in the buyer's date-τ value. Intuitively, this is because earlier participation gives the buyer access to higher quality levels, for the same evolution of his values (assuming these values are reported truthfully). Like Condition (ii), Condition (i) is a kind of monotonicity condition — it implies monotonicity of the allocations in the participation date. Like Condition (ii), it is somewhat stronger than required, but it is simple to state and a natural analogue to the condition of Proposition 1 for the two-value case considered above.

At least when the conditions of Proposition 3 hold, we are able to confirm the findings in Corollary 1. If $\frac{\beta_\tau}{\rho_\tau}$ is increasing in τ, then qualities are distorted further below the first-best level at later dates and the buyer expects less rent conditional on his value at arrival. A generalized "principle of vanishing distortions" also applies, provided that the impulse response functions $J_\tau^t\left(\theta_{\tau,\tau}^t\right)$ vanish uniformly over time.

We next provide examples of processes for which Proposition 3 is satisfied.

Example 1 *Let $\bar{\theta} > 0$, let F_{In} be the uniform distribution on $\left[0, \bar{\theta}\right]$, and let ϕ be a positive scalar. For each τ and each $t > \tau$, let $\tilde{\theta}_{\tau,t} = \bar{\theta}\left(1 - \tilde{\varepsilon}_{t-\tau} e^{-\phi\theta_{\tau,t-1}}\right)$, where $\tilde{\varepsilon}_{t-\tau}$ is a random variable distributed uniformly on the unit interval. For each τ, each $t > \tau$, and each $\theta_{\tau,\tau}^t$,*

$$m_\tau^t\left(\theta_{\tau,\tau}^t\right) = \theta_{\tau,t} - \phi^{t-\tau}\Pi_{s=\tau}^t\left(\bar{\theta} - \theta_s\right)\left(\frac{\beta_\tau}{\rho_\tau} + 1\right). \tag{14}$$

The conditions of Proposition 3 are satisfied provided that, for all τ,

$$\phi\bar{\theta}\left(\frac{\beta_\tau}{\rho_\tau} + 1\right) \leq \frac{\beta_{\tau+1}}{\rho_{\tau+1}} + 1. \tag{15}$$

Example 1 is notable in that it typically includes a wide class of distributions for the arrival date. If $\phi\bar{\theta} \leq 1$, for instance, then it is enough that $\frac{\beta_\tau}{\rho_\tau}$ is non-decreasing in τ, which is the condition we emphasized for the two-value case. More generally, Condition (i) of Proposition 3 is more likely to hold if the process is not too persistent. In the above example, the impulse response function is given, for any dates τ, and $t > \tau$, by $J_\tau^t\left(\theta_{\tau,\tau}^t\right) = \phi^{t-\tau}\Pi_{s=\tau+1}^t\left(\bar{\theta} - \theta_{\tau,s}\right)$; thus ϕ is a parameter which indexes the persistence of the process and the condition holds more easily whenever ϕ is small.

An important class of examples in the literature, beginning with Besanko (1985), concerns autoregressive processes. Suppose the buyer's value evolves according to an autoregressive process, with $\tilde{\theta}_{\tau,t} = \gamma\theta_{\tau,t-1} + \tilde{\varepsilon}_t$ for some $\gamma \in (0,1]$ and $\tilde{\varepsilon}_t$ an independently distributed "shock". In this case, Condition (ii) is often straightforward to check, while Condition (i) is more difficult, unless restrictive assumptions are made on the distribution of arrivals. Condition (i) is easier to check when there are two possible arrival dates, however, as in the following example.

Example 2 *Suppose that $T = 2$. Let F_{In} be the uniform distribution on $\Theta = \left[\underline{\theta}, \bar{\theta}\right]$, which determines the distribution of $\tilde{\theta}_{1,1}$ and $\tilde{\theta}_{2,2}$. Let $\gamma \in (0,1]$ and let G be a continuously differentiable c.d.f. on $\left[\underline{\theta}\,(1-\gamma), \bar{\theta}\,(1-\gamma)\right]$. Suppose that $\tilde{\theta}_{1,2}$ is distributed according to $\gamma\theta_{1,1} + \tilde{\varepsilon}$, where $\tilde{\varepsilon}$ is distributed according to G. Then $m_1^1\left(\theta_{1,1}\right) = \theta_{1,1} - \frac{1-F_{In}(\theta_{1,1})}{f_{In}(\theta_{1,1})}$, $m_1^2\left(\theta_{1,1}^2\right) = \theta_{1,2} - \gamma\frac{1-F_{In}(\theta_{1,1})}{f_{In}(\theta_{1,1})}$, and $m_2^2\left(\theta_{2,2}\right) = \theta_{2,2} - \frac{1-F_{In}(\theta_{2,2})}{f_{In}(\theta_{2,2})} - \frac{\rho_1}{\rho_2}\frac{1-G(\theta_{2,2}-\gamma\underline{\theta})}{f_{In}(\theta_{2,2})}$. Then both conditions of Proposition 3 are satisfied.*

While verifying the conditions in Proposition 3 can be difficult, certain qualitative properties of the optimal mechanism are quite robust. Indeed, under fairly general conditions (requiring neither of the Conditions (i) or (ii) of Proposition 3), we can establish a partial analogue of Corollary 2. This describes how the agent fares for all sufficiently late participation dates.

Proposition 4 *Suppose that $T = +\infty$, with $\rho_\tau > 0$ for all τ. Then the following are true of an optimal mechanism:*

(i) $V_{\tau,\tau}(\underline{\theta};\emptyset)$ converges to zero with τ.

(ii) If, in addition, $F_{T\tau}(\cdot|\underline{\theta})$ has full support on Θ, then the buyer's expected rents conditional on participation at date τ, $\mathbb{E}\left[V_{\tau,\tau}\left(\tilde{\theta}_{\tau,\tau};\emptyset\right)\right]$, converge to zero with τ.

For Part (i) of the proposition, the intuition is the familiar one: reducing rents at later dates allows the seller to reduce rents also at all earlier dates, so the seller does well to pick $V_{\tau,\tau}(\underline{\theta};\emptyset)$ close to zero for large τ. However, this does not necessarily imply that $V_{\tau,\tau}(\theta_{\tau,\tau};\emptyset)$ should vanish with τ for all $\theta_{\tau,\tau}$. In particular, one can find processes with $\bar{\theta}(\underline{\theta}) < \bar{\theta}$ such that the buyer continues to expect positive rent upon arrival with a value larger than $\bar{\theta}(\underline{\theta})$ under an optimal mechanism. Intuitively, the reason is that permitting the buyer a large rent for high initial values need not create a valuable option for the buyer when he arrives in the previous period. For instance, such high values might only be obtained at the buyer's arrival date. The full-support assumption in Part (ii) of the proposition guarantees that this does not happen.

Proposition 4 also has implications for optimal qualities, which can be understood by examining Lemma 4. In particular, note that

$$V_{\tau,\tau}^{\Omega}\left(\theta_{\tau,\tau};\emptyset\right) = V_{\tau,\tau}^{\Omega}\left(\underline{\theta};\emptyset\right) + \int_{\underline{\theta}}^{\theta_{\tau,\tau}} \mathbb{E}\left[\sum_{s=\tau}^{\infty} \delta^{s-\tau} J_{\tau}^{s}\left(\tilde{\theta}_{\tau,\tau}^{s}\right) q_{\tau,s}^{*}\left(\tilde{\theta}_{\tau,\tau}^{s}\right) |\tilde{\theta}_{\tau,\tau} = r\right] dr$$

under an optimal mechanism with allocation rule $\left(q^*_{\tau,t}\right)_{1\leq\tau\leq t}$. Hence, if $F_{T r}(\cdot|\underline{\theta})$ has full support on Θ, the observation that $\mathbb{E}\left[V_{\tau,\tau}\left(\tilde{\theta}_{\tau,\tau};\emptyset\right)\right]$ vanishes with τ (Part (ii) of Proposition 4) implies that the allocations $q^*_{\tau,s}\left(\theta^s_{\tau,\tau}\right)$ cannot be too large at histories $\theta^s_{\tau,\tau}$ such that the impulse responses $J^s_\tau\left(\theta^s_{\tau,\tau}\right)$ are large, except perhaps with small probability.[7]

5 Relation to other literature

This section provides further discussion of the connections to several papers. First, note that an earlier working paper, Garrett (2011), was the first to formally analyze a dynamic mechanism design setting where an agent arrives over time and where the agent's values evolve stochastically (and where both arrival time and valuations are the agent's private information). A key aim of the present paper was to summarize insights from the earlier (unpublished) work, but in a simplified setting.[8] Subsequent work by Deb and Said (2015) provides an analysis of a two-period setting, where a unit is allocated in the second period and there is no competition among buyers. Deb and Said solve the case where the seller fully commits (as in the present work), but then focuses on relaxing this commitment ability. That the seller allocates a single homogeneous unit to a single buyer simplifies the analysis, facilitating a characterization of the optimal mechanism under quite weak restrictions on the distribution of buyer information and values (in particular, see the full-commitment case). The present paper instead considers variable quality in a repeated Mussa-Rosen framework. The approach in this paper necessarily differs from Deb and Said and (as noted in the Introduction) could readily be applied in other problems (including multi-agent settings). Ely, Garrett and Hinnosaar (forthcoming) also provide an analysis of a two-period problem, with allocation of the good in the second period. The focus there, however, is on a restricted "simple" mechanism (where early ticket sales are made at a single price, but auctions are permitted to reallocate capacity). Unlike the two-period settings of Deb and Said and Ely, Garrett and Hinnosaar, the present paper analyzes an arbitrary horizon length. It therefore elucidates how optimal mechanisms evolve when agent participation can take place over longer horizons, and shows how the distortions in optimal mechanisms tend to accumulate over time, so that agents who arrive later receive more distorted allocations. Studying longer horizons allowed us to examine issues such as the limiting behavior of the mechanism as the arrival time becomes arbitrarily late, as well as the applicability of the so-called "principle of vanishing distortions" for relationships that have lasted a sufficiently long time.

[7] Conversely, if the buyer's values are not very persistent, then qualities may not be very distorted after the early periods of the relationship. In the extreme case, where the buyer's values are independently distributed across time, we have $J^s_\tau\left(\theta^s_{\tau,\tau}\right) = 0$ whenever $s > \tau$. The optimal allocation then coincides with the efficient allocation at every date after the arrival date τ (i.e., $q^*_{\tau,s}\left(\theta^s_{\tau,\tau}\right) = \theta_{\tau,s}$ for all $s > \tau$ and all $\theta^s_{\tau,\tau}$).

[8] As noted in the Introduction, the earlier paper considered a durable goods setting. The dynamic optimization problem there is more complex, motivating the simplified treatment in the present version.

Perhaps the most important antecedents to Garrett (2011), and hence the present paper, are Deb (2011, 2014) and Nocke, Peitz and Rosar (2011). Deb studies a seller's optimal price path in an infinite-horizon setting where a buyer arrives at a fixed date (date zero), and his value changes at a single random time. Deb finds that the optimal price path often features low introductory pricing. Given the restriction to a price path, a buyer often chooses not to participate in the mechanism at first instance; that is, he chooses to delay his purchase decision. The seller may choose a higher price at later dates (after date zero) precisely to deter delay in purchasing. Nocke, Peitz and Rosar study a two-period model where the buyer learns his value only at the second date. They also find that introductory pricing can be optimal (again, this can reflect the seller's aim to punish delayed purchase). The optimal price path in their paper turns out to implement also the optimal mechanism (as chosen without any restrictions). Notably, this means that the seller sometimes finds it optimal to induce participation at a date after the buyer is initially available, although by the revelation principle she could achieve the same outcome by always inducing participation at the initial date. In our setting, the seller instead typically finds it *strictly* more profitable to induce buyer participation in the mechanism at the first possible instant. Participation then occurs at different dates in our optimal mechanism only because the buyer's arrival date is random/heterogeneous.

Other papers also now highlight the value to a seller of deterring delayed purchase. Garrett (forthcoming) studies the optimal price path in an infinite-horizon setting where buyers arrive over time, and where values then change randomly over time. Armstrong and Zhou (2015) study commitments a seller may make to deter buyers from searching for a better product and then returning to purchase. While these papers focus on particular applications and selling formats, the present paper focuses on developing a mechanism design approach that can be applied quite generally in settings where agents arrive over time and have preferences that change randomly.

6 Conclusions

This paper has considered dynamic mechanism design in a setting where buyers arrive over time and where their preferences evolve stochastically. We showed how it is often possible to fully characterize the optimal mechanism. The key finding, which applies across the canonical two-type setting, and the setting with a continuum of values, is that a late participant is punished in that he faces tougher terms of trade and therefore purchases lower qualities and receives less rent. Early arrivers fare better, and buyers earn positive expected rents even if their values are equal to their lowest. Although later arrivals receive less efficient allocations for longer, the "principle of vanishing distortions" by which allocations converge to first-best levels with time in the relationship continues to apply for appropriate restrictions on the process for values. These findings can be expected to have relevance not only for buyer and seller relationships, but for a broad class of agency relationships, including dynamic regulation, dynamic employment relationships and dynamic public finance. Such applications may be fruitful areas for future work.

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Appendix: Proofs of results

Proof of Lemma 1. Consider a buyer who at date t has reported $\hat{\theta}_{\tau,\tau}^{t-1}$ from date τ up to date $t-1$. That the buyer must be willing to report truthfully a date$-t$ value θ_H implies

$$\begin{aligned} V_{\tau,t}^{\Omega}\left(\theta_H;\hat{\theta}_{\tau,\tau}^{t-1}\right) - V_{\tau,t}^{\Omega}\left(\theta_L;\hat{\theta}_{\tau,\tau}^{t-1}\right) \geq\ & (\theta_H-\theta_L)\, q_{\tau,t}\left(\hat{\theta}_{\tau,\tau}^{t-1},\theta_L\right) \\ & +\delta\left(\alpha_H-\alpha_L\right)\left(V_{\tau,t+1}^{\Omega}\left(\theta_H;\hat{\theta}_{\tau,\tau}^{t-1},\theta_L\right) - V_{\tau,t+1}^{\Omega}\left(\theta_L;\hat{\theta}_{\tau,\tau}^{t-1},\theta_L\right)\right). \end{aligned} \tag{16}$$

Suppose that

$$\begin{aligned} & V_{\tau,t}^{\Omega}\left(\theta_H;\hat{\theta}_{\tau,\tau}^{t-1}\right) - V_{\tau,t}^{\Omega}\left(\theta_L;\hat{\theta}_{\tau,\tau}^{t-1}\right) \\ \geq\ & (\theta_H-\theta_L)\sum_{s=t}^{t'}\delta^{s-t}\left(\alpha_H-\alpha_L\right)^{s-t} q_{\tau,s}\left(\hat{\theta}_{\tau,\tau}^{t-1},\theta_L^{s-t+1}\right) \\ & +\delta^{t'-t+1}\left(\alpha_H-\alpha_L\right)^{t'-t+1}\left(V_{\tau,t'+1}^{\Omega}\left(\theta_H;\hat{\theta}_{\tau,\tau}^{t-1},\theta_L^{t'-t+1}\right) - V_{\tau,t'+1}^{\Omega}\left(\theta_L;\hat{\theta}_{\tau,\tau}^{t-1},\theta_L^{t'-t+1}\right)\right) \end{aligned}$$

holds for some $t'>t$. Using (16) to substitute for the final term then yields

$$\begin{aligned} & V_{\tau,t}^{\Omega}\left(\theta_H;\hat{\theta}_{\tau,\tau}^{t-1}\right) - V_{\tau,t}^{\Omega}\left(\theta_L;\hat{\theta}_{\tau,\tau}^{t-1}\right) \\ \geq\ & (\theta_H-\theta_L)\sum_{s=t}^{t''}\delta^{s-t}\left(\alpha_H-\alpha_L\right)^{s-t} q_{\tau,s}\left(\hat{\theta}_{\tau,\tau}^{t-1},\theta_L^{s-t+1}\right) \\ & +\delta^{t''-t+1}\left(\alpha_H-\alpha_L\right)^{t''-t+1}\left(V_{\tau,t''+1}^{\Omega}\left(\theta_H;\hat{\theta}_{\tau,\tau}^{t-1},\theta_L^{t''-t+1}\right) - V_{\tau,t''+1}^{\Omega}\left(\theta_L;\hat{\theta}_{\tau,\tau}^{t-1},\theta_L^{t''-t+1}\right)\right) \end{aligned}$$

for $t''=t'+1$. The result then follows by induction and (for the case of $T=+\infty$) the observation that, in an incentive-compatible mechanism, $V_{\tau,s+1}^{\Omega}\left(\theta_H;\hat{\theta}_{\tau,\tau}^{t-1},\theta_L^{s-t+1}\right) - V_{\tau,s+1}^{\Omega}\left(\theta_L;\hat{\theta}_{\tau,\tau}^{t-1},\theta_L^{s-t+1}\right)$ must be uniformly bounded for all $s \geq t$. ■

Proof of Lemma 2. For an incentive-compatible mechanism Ω and any date τ, the buyer's expected payoff must satisfy

$$\begin{aligned} V_{\tau,\tau}^{\Omega}\left(\theta_L;\emptyset\right) &\geq \delta\left(\left(1-\alpha_L\right)V_{\tau+1,\tau+1}^{\Omega}\left(\theta_L;\emptyset\right)+\alpha_L V_{\tau+1,\tau+1}^{\Omega}\left(\theta_H;\emptyset\right)\right) \\ &= \delta\left(V_{\tau+1,\tau+1}^{\Omega}\left(\theta_L;\emptyset\right)+\alpha_L\left(V_{\tau+1,\tau+1}^{\Omega}\left(\theta_H;\emptyset\right)-V_{\tau+1,\tau+1}^{\Omega}\left(\theta_L;\emptyset\right)\right)\right) \\ &\geq \delta\left(V_{\tau+1,\tau+1}^{\Omega}\left(\theta_L;\emptyset\right)+\alpha_L\left(\theta_H-\theta_L\right)\sum_{s=\tau+1}^{T}\delta^{s-\tau-1}\left(\alpha_H-\alpha_L\right)^{s-\tau-1}q_{\tau+1,s}\left(\theta_L^{s-\tau}\right)\right), \end{aligned}$$

where the final equality follows from Lemma 1. The same inequality holds also for $V^{\Omega}_{\tau+1,\tau+1}(\theta_L;\emptyset)$. Hence,

$$\begin{aligned}V^{\Omega}_{\tau,\tau}(\theta_L;\emptyset) \geq\ & \delta\alpha_L(\theta_H-\theta_L)\sum_{s=\tau+1}^{T}\delta^{s-\tau-1}(\alpha_H-\alpha_L)^{s-\tau-1}q_{\tau+1,s}\left(\theta_L^{s-\tau}\right)\\ & +\delta^2\alpha_L(\theta_H-\theta_L)\sum_{s=\tau+2}^{T}\delta^{s-\tau-2}(\alpha_H-\alpha_L)^{s-\tau-2}q_{\tau+2,s}\left(\theta_L^{s-\tau-1}\right)\\ & +\delta^2 V^{\Omega}_{\tau+2,\tau+2}(\theta_L;\emptyset).\end{aligned}$$

The result then follows from induction, and the fact that $V^{\Omega}_{\tau',\tau'}(\theta_L;\emptyset)$ remains bounded uniformly over $\tau'>\tau$. ■

Proof of Lemma 3. First note that the buyer's expected rent is given by

$$\begin{aligned}&\sum_{\tau=1}^{T}\delta^{\tau-1}\rho_\tau\left(\mu V^{\Omega}_{\tau,\tau}(\theta_H;\emptyset)+(1-\mu)V^{\Omega}_{\tau,\tau}(\theta_L;\emptyset)\right)\\ =\ &\sum_{\tau=1}^{T}\delta^{\tau-1}\rho_\tau V^{\Omega}_{\tau,\tau}(\theta_L;\emptyset)+\mu\sum_{\tau=1}^{T}\delta^{\tau-1}\rho_\tau\left(V^{\Omega}_{\tau,\tau}(\theta_H;\emptyset)-V^{\Omega}_{\tau,\tau}(\theta_L;\emptyset)\right).\end{aligned}$$

The first term reflects the rent that the buyer expects to earn even if his value is low at the arrival date, while the second term reflects the additional rent he expects if his value is instead high. A lower bound for the first term is available from Lemma 2:

$$\sum_{\tau=1}^{T}\delta^{\tau-1}\rho_\tau V^{\Omega}_{\tau,\tau}(\theta_L;\emptyset)\geq\alpha_L(\theta_H-\theta_L)\sum_{\tau=2}^{T}\sum_{s=\tau}^{T}\delta^{s-1}\beta_\tau(\alpha_H-\alpha_L)^{s-\tau}q_{\tau,s}\left(\theta_L^{s-\tau+1}\right).$$

A lower bound for the second term is available from simply substituting the expression in Lemma 1:

$$\mu\sum_{\tau=1}^{T}\delta^{\tau-1}\rho_\tau\left(V^{\Omega}_{\tau,\tau}(\theta_H;\emptyset)-V^{\Omega}_{\tau,\tau}(\theta_L;\emptyset)\right)\geq\mu(\theta_H-\theta_L)\sum_{\tau=1}^{T}\rho_\tau\left(\sum_{s=\tau}^{T}\delta^{s-1}(\alpha_H-\alpha_L)^{s-\tau}q_{\tau,s}\left(\theta_L^{s-\tau+1}\right)\right).$$

Therefore, the rents that a buyer is expected to earn must be at least

$$\sum_{\tau=1}^{T}\sum_{s=\tau}^{T}\delta^{s-1}(\alpha_L\beta_\tau+\mu\rho_\tau)(\theta_H-\theta_L)(\alpha_H-\alpha_L)^{s-\tau}q_{\tau,s}\left(\theta_L^{s-\tau+1}\right).$$

The expression for profits in the lemma is then simply the expected surplus less the lower bound on buyer expected rents. ■

Proof of Proposition 1. The allocations $\mathbf{q}^*=\left(q^*_{\tau,t}\right)_{1\leq\tau\leq t}$ are chosen to maximize (4). (A unique optimum exists by convexity of the cost function $c(\cdot)$.) It remains to verify the existence of a system of transfers $\mathbf{p}$ which implements $\mathbf{q}^*$ as part of an incentive-compatible mechanism. To this end, we begin by specifying the payoff that the buyer expects from truthful reporting at each date t

following any history of reports $\hat{\theta}_{\tau,\tau}^{t-1}$ from date τ. We choose these payoffs so that the inequalities (1) and (3) hold with equality, which in turn implies that the buyer's expected rents are as small as possible in an incentive-compatible and individually-rational mechanism implementing $\mathbf{q}^*$. This means that expected profits are equal to the expression in (4).

There is still much freedom in how payoffs are spread across time. One possible specification is as follows: At each date τ of first reporting

$$V_{\tau,\tau}^{\Omega}\left(\theta_L;\emptyset\right)=\alpha_L\left(\theta_H-\theta_L\right)\sum_{i=1}^{T-\tau}\delta^i\left(\sum_{s=\tau+i}^{T}\delta^{s-\tau-i}\left(\alpha_H-\alpha_L\right)^{s-\tau-i}q_{\tau+i,s}^{*}\left(\theta_L^{s-\tau-i+1}\right)\right).$$

For each $t>\tau$, and each history of reports $\hat{\theta}_{\tau,\tau}^{t-1}$, $V_{\tau,t}^{\Omega}\left(\theta_L;\hat{\theta}_{\tau,\tau}^{t-1}\right)=0$. For each τ, each $t\geq\tau$, and each $\hat{\theta}_{\tau,\tau}^{t-1}$,

$$V_{\tau,t}^{\Omega}\left(\theta_H;\hat{\theta}_{\tau,\tau}^{t-1}\right)=V_{\tau,t}^{\Omega}\left(\theta_L;\hat{\theta}_{\tau,\tau}^{t-1}\right)+\left(\theta_H-\theta_L\right)\sum_{s=t}^{T}\delta^{s-t}\left(\alpha_H-\alpha_L\right)^{s-t}q_{\tau,s}^{*}\left(\hat{\theta}_{\tau,\tau}^{t-1},\theta_L^{s-t+1}\right).$$

Next, one can choose prices to ensure that these payoffs are realized if the buyer reports truthfully. This is achieved if, for each τ, each $t\geq\tau$ and each $\hat{\theta}_{\tau,\tau}^{t-1}$, we let

$$p_{\tau,t}^{*}\left(\hat{\theta}_{\tau,\tau}^{t-1},\theta_{\tau,t}\right)=\theta_{\tau,t}q_{\tau,t}^{*}\left(\hat{\theta}_{\tau,\tau}^{t-1},\theta_{\tau,t}\right)-V_{\tau,t}^{\Omega}\left(\theta_{\tau,t};\hat{\theta}_{\tau,\tau}^{t-1}\right)+\delta\mathbb{E}\left[V_{\tau,t+1}^{\Omega}\left(\tilde{\theta}_{\tau,t+1};\hat{\theta}_{\tau,\tau}^{t-1},\theta_{\tau,t}\right)|\theta_{\tau,t}\right].$$

Now, we wish to check that the mechanism $\langle\mathbf{q}^*,\mathbf{p}^*\rangle$, with $\mathbf{p}^*=\left(p_{\tau,t}^*\right)_{1\leq\tau\leq t}$ is incentive compatible. Two kinds of incentive constraints must be checked. First, conditional on the buyer having reported to the mechanism, he must be willing to report his values truthfully. Second, he must be willing to participate in the mechanism and report to it immediately on the date of his arrival.

Truthful reporting of values. By the "one-shot deviation principle" of Blackwell (1965), it is enough to check that one-shot deviations from truth-telling are never optimal, for any history of past reports. Because the process is first-order Markov, the payoffs available to the buyer at any date t depend only on his date$-t$ value $\theta_{\tau,t}$, and the past reports $\hat{\theta}_{\tau,\tau}^{t-1}$, and not on any previous values. Verifying that the buyer does not profit from a one-shot deviation when his value is high amounts to verifying (16), which holds by construction. Verifying that the buyer does not profit from a one-shot deviation when his value is low amounts to checking

$$\begin{aligned}V_{\tau,t}^{\Omega}\left(\theta_L;\hat{\theta}_{\tau,\tau}^{t-1}\right)\geq{}&V_{\tau,t}^{\Omega}\left(\theta_H;\hat{\theta}_{\tau,\tau}^{t-1}\right)-\left(\theta_H-\theta_L\right)q_{\tau,t}^{*}\left(\hat{\theta}_{\tau,\tau}^{t-1},\theta_H\right)\\&-\delta\left(\alpha_H-\alpha_L\right)\left(V_{\tau,t+1}^{\Omega}\left(\theta_H;\hat{\theta}_{\tau,\tau}^{t-1},\theta_H\right)-V_{\tau,t+1}^{\Omega}\left(\theta_L;\hat{\theta}_{\tau,\tau}^{t-1},\theta_H\right)\right).\end{aligned}$$

That (1) holds with equality at all histories implies that this is equivalent to

$$\begin{aligned}
& (\theta_H - \theta_L) \sum_{s=t}^{T} \delta^{s-t} (\alpha_H - \alpha_L)^{s-t} q^*_{\tau,s} \left(\hat{\theta}^{t-1}_{\tau,\tau}, \theta_L^{s-t+1}\right) \\
\leq \; & (\theta_H - \theta_L) q^*_{\tau,t} \left(\hat{\theta}^{t-1}_{\tau,\tau}, \theta_H\right) + \delta (\alpha_H - \alpha_L) \left(V^{\Omega}_{\tau,t+1}\left(\theta_H; \hat{\theta}^{t-1}_{\tau,\tau}, \theta_H\right) - V^{\Omega}_{\tau,t+1}\left(\theta_L; \hat{\theta}^{t-1}_{\tau,\tau}, \theta_H\right)\right) \\
= \; & (\theta_H - \theta_L) \sum_{s=t}^{T} \delta^{s-t} (\alpha_H - \alpha_L)^{s-t} q^*_{\tau,s} \left(\hat{\theta}^{t-1}_{\tau,\tau}, \theta_H, \theta_L^{s-t}\right).
\end{aligned}$$

This is satisfied because, for all τ, t and s, with $\tau \leq t \leq s$, and all $\hat{\theta}^{t-1}_{\tau,\tau}$, $q^*_{\tau,s}\left(\hat{\theta}^{t-1}_{\tau,\tau}, \theta_H, \theta_L^{s-t}\right) \geq q^*_{\tau,s}\left(\hat{\theta}^{t-1}_{\tau,\tau}, \theta_L^{s-t+1}\right)$.

Timely participation. The above implies that, if the buyer participates at date τ, he then reports his values truthfully from then on, and therefore expects to earn the payoffs specified above. Since transition probabilities do not depend on the buyer's arrival date to the market, the buyer's problem of whether to participate is identical irrespective of his true arrival date. Specifying that the buyer participates at every opportunity, we can then check that the buyer does not gain from one-shot deviations, i.e. from delaying participation. This follows when the buyer's value is low by (2), which is satisfied by construction.

For a high value, we need to check

$$V^{\Omega}_{\tau,\tau}(\theta_H; \emptyset) \geq \delta \left((1-\alpha_H) V^{\Omega}_{\tau+1,\tau+1}(\theta_L; \emptyset) + \alpha_H V^{\Omega}_{\tau+1,\tau+1}(\theta_H; \emptyset)\right).$$

This is equivalent to

$$(\theta_H - \theta_L) q^*_{\tau,\tau}(\theta_L) + (\theta_H - \theta_L) \sum_{s=\tau+1}^{T} \delta^{s-\tau} (\alpha_H - \alpha_L)^{s-\tau} \left(q^*_{\tau,s}\left(\theta_L^{s-\tau+1}\right) - q^*_{\tau+1,s}\left(\theta_L^{s-\tau}\right)\right) \geq 0. \quad (17)$$

It is readily checked that, for all $\tau \leq T-1$, $m^{\tau+1}_{\tau+1}(\theta_L) \leq m^{\tau+1}_{\tau}\left(\theta_L^2\right)$ implies $m^s_{\tau+1}\left(\theta_L^{s-\tau}\right) \leq m^s_{\tau}\left(\theta_L^{s-\tau+1}\right)$ for all $s \geq \tau+1$. Therefore, $q^*_{\tau+1,s}\left(\theta_L^{s-\tau+1}\right) \leq q^*_{\tau,s}\left(\theta_L^{s-\tau+1}\right)$ for all $s \geq \tau+1$, so that (17) is indeed satisfied. ■

Proof of Corollary 1. The first part follows directly from the qualities specified in Proposition 1. The second part follows using these optimal qualities and the fact that the buyer payoffs at the participation/arrival date $V_{\tau,\tau}(\theta_{\tau,\tau}; \emptyset)$ satisfy the inequalities (1) and (3) with equality. ■

Proof of Corollary 2. The first part follows directly from the qualities specified in Proposition 1. The implication for buyer rents follows using the optimal qualities and the fact that the buyer payoffs at the participation/arrival date $V_{\tau,\tau}(\theta_{\tau,\tau}; \emptyset)$ satisfy the inequalities (1) and (3) with equality. ■

Proof of Corollary 3. This follows immediately from the qualities specified in Proposition 1. ■

Proof of Proposition 2. The lower bound (3) in Lemma 2 becomes

$$V^{\Omega}_{\tau,\tau}\left(\theta_L;\emptyset\right) \geq \alpha^W_L \left(\theta_H - \theta_L\right) \sum_{i=1}^{T-\tau} \sum_{s=\tau+i}^{T} \delta^{s-\tau} \left(\alpha_H - \alpha_L\right)^{s-\tau-i} q_{\tau+i,s}\left(\theta_L^{s-\tau-i+1}\right).$$

Together with the inequalities in Lemma 1 one obtains a lower bound on buyer expected rents. This allows us to derive the upper bound on expected profits

$$\mathbb{E}\left[\sum_{s=\tilde{\tau}}^{T} \delta^{s-1}\left(m^{W,s}_{\tilde{\tau}}\left(\tilde{\theta}^s_{\tilde{\tau},\tilde{\tau}}\right) q_{\tilde{\tau},s}\left(\tilde{\theta}^s_{\tilde{\tau},\tilde{\tau}}\right) - c\left(q_{\tilde{\tau},s}\left(\tilde{\theta}^s_{\tilde{\tau},\tilde{\tau}}\right)\right)\right)\right],$$

with virtual values $m^{W,s}_{\tau}$ given in the present result. One then chooses qualities to maximize this expression and then proposes an appropriate implementation as in the proof of Proposition 1. In particular, one should specify payments such that all incentive (and individual rationality) constraints bind, and can than specify that, for each $t > \tau$, and each history of reports $\hat{\theta}^{t-1}_{\tau,\tau}$, $V^{\Omega}_{\tau,t}\left(\theta_L;\hat{\theta}^{t-1}_{\tau,\tau}\right) = 0$.

Verifying that the buyer, having chosen to participate in the mechanism, is willing to report all values truthfully follows the same steps as in the proof of Proposition 1. It then remains to verify the buyer's willingness to participate at his arrival date. By construction, he is indifferent to doing so when his value is low. When his value is high, it is enough to verify that

$$\begin{aligned} &\left(\theta_H - \theta_L\right) \sum_{s=\tau}^{T} \delta^{s-\tau} \left(\alpha_H - \alpha_L\right)^{s-\tau} q^W_{\tau,s}\left(\theta_L^{s-\tau+1}\right) \\ \geq\ & \left(\alpha^W_H - \alpha^W_L\right)\left(\theta_H - \theta_L\right) \sum_{s=\tau+1}^{T} \delta^{s-\tau} \left(\alpha_H - \alpha_L\right)^{s-\tau-1} q^W_{\tau+1,s}\left(\theta_L^{s-\tau}\right). \end{aligned}$$

That this is satisfied follows because $\alpha^W_H - \alpha^W_L \leq 1$, and because $\frac{\beta_\tau}{\rho_\tau}$ is increasing in τ. The latter guarantees that $q^W_{\tau,s}\left(\theta_L^{s-\tau+1}\right) \geq q^W_{\tau+1,s+1}\left(\theta_L^{s-\tau+1}\right)$ for all $s \leq T-1$. ■

Proof of Lemma 4. This follows immediately from Theorem 1 of Pavan, Segal and Toikka (2014). ■

Proof of Lemma 5. By (10), for any τ,

$$\begin{aligned}
V_{\tau,\tau}^{\Omega}\left(\underline{\theta};\emptyset\right) &\geq \delta V_{\tau+1,\tau+1}^{\Omega}\left(\underline{\theta};\emptyset\right) \\
&+\mathbb{E}\left[\int_{\underline{\theta}}^{\tilde{\theta}_{\tau,\tau+1}}\mathbb{E}\left[\sum_{t=\tau+1}^{T}\delta^{t-\tau}J_{\tau+1}^{t}\left(\tilde{\theta}_{\tau+1,\tau+1}^{t}\right)q_{\tau+1,t}\left(\tilde{\theta}_{\tau+1,\tau+1}^{t}\right)\mid\tilde{\theta}_{\tau+1,\tau+1}=r\right]dr\mid\tilde{\theta}_{\tau,\tau}=\underline{\theta}\right] \\
&= \delta V_{\tau+1,\tau+1}^{\Omega}\left(\underline{\theta};\emptyset\right) \\
&+\mathbb{E}\left[\frac{1-F_{Tr}\left(\tilde{\theta}_{\tau,\tau+1}|\underline{\theta}\right)}{f_{Tr}\left(\tilde{\theta}_{\tau,\tau+1}|\underline{\theta}\right)}\sum_{t=\tau+1}^{T}\delta^{t-\tau}J_{\tau+1}^{t}\left(\tilde{\theta}_{\tau+1,\tau+1}^{t}\right)q_{\tau+1,t}\left(\tilde{\theta}_{\tau+1,\tau+1}^{t}\right)\mid\tilde{\theta}_{\tau,\tau}=\underline{\theta}\right] \\
&= \delta V_{\tau+1,\tau+1}^{\Omega}\left(\underline{\theta};\emptyset\right) \\
&+\mathbb{E}\left[\frac{1-F_{Tr}\left(\tilde{\theta}_{\tau+1,\tau+1}|\underline{\theta}\right)}{f_{In}\left(\tilde{\theta}_{\tau+1,\tau+1}\right)}\sum_{t=\tau+1}^{T}\delta^{t-\tau}J_{\tau+1}^{t}\left(\tilde{\theta}_{\tau+1,\tau+1}^{t}\right)q_{\tau+1,t}\left(\tilde{\theta}_{\tau+1,\tau+1}^{t}\right)\right],
\end{aligned}$$

where the first equality follows from integration by parts and the second by a simple rearrangement. Iterating then yields (11). ■

Proof of Lemma 6. By Lemmas 4 and 5, the buyer's expected rent conditional on arriving at date τ is at least

$$\begin{aligned}
&\sum_{i=1}^{T-\tau}\mathbb{E}\left[\frac{1-F_{Tr}\left(\tilde{\theta}_{\tau+i,\tau+i}|\underline{\theta}\right)}{f_{In}\left(\tilde{\theta}_{\tau+i,\tau+i}\right)}\sum_{t=\tau+i}^{T}\delta^{t-\tau}J_{\tau+i}^{t}\left(\tilde{\theta}_{\tau+i,\tau+i}^{t}\right)q_{\tau+i,t}\left(\tilde{\theta}_{\tau+i,\tau+i}^{t}\right)\right] \\
&+\mathbb{E}\left[\int_{\underline{\theta}}^{\tilde{\theta}_{\tau,\tau}}\mathbb{E}\left[\sum_{s=\tau}^{T}\delta^{s-\tau}J_{\tau}^{s}\left(\tilde{\theta}_{\tau,\tau}^{s}\right)q_{\tau,s}\left(\tilde{\theta}_{\tau,\tau}^{s}\right)|\tilde{\theta}_{\tau,\tau}=r\right]dr\right].
\end{aligned}$$

For each τ, integrate the second term by parts and subtract the full expression for buyer expected rents from the expected surplus. Taking expectations over the arrival date τ then yields the result. ■

Proof of Proposition 3. Since the qualities $q_{\tau,t}^{*}$ are chosen to maximize (12), we need only to provide an incentive-compatible mechanism which implements them. As for the proof of Proposition 1, we begin by specifying the buyer's expected payoffs that the mechanism is to deliver the buyer when he reports his values truthfully. For all τ and all $\theta_{\tau,\tau}$, let

$$\begin{aligned}
V_{\tau,\tau}^{\Omega}\left(\theta_{\tau,\tau};\emptyset\right) &= \sum_{i=1}^{T-\tau}\mathbb{E}\left[\frac{1-F_{Tr}\left(\tilde{\theta}_{\tau+i,\tau+i}|\underline{\theta}\right)}{f_{In}\left(\tilde{\theta}_{\tau+i,\tau+i}\right)}\sum_{t=\tau+i}^{T}\delta^{t-\tau}J_{\tau+i}^{t}\left(\tilde{\theta}_{\tau+i,\tau+i}^{t}\right)q_{\tau+i,t}^{*}\left(\tilde{\theta}_{\tau+i,\tau+i}^{t}\right)\right] \\
&+\int_{\underline{\theta}}^{\theta_{\tau,\tau}}\mathbb{E}\left[\sum_{s=\tau}^{T}\delta^{s-\tau}J_{\tau}^{s}\left(\tilde{\theta}_{\tau,\tau}^{s}\right)q_{\tau,s}^{*}\left(\tilde{\theta}_{\tau,\tau}^{s}\right)|\tilde{\theta}_{\tau,\tau}=r\right]dr.
\end{aligned}$$

For all τ, all $t > \tau$, and all $\left(\hat{\theta}_{\tau,\tau}^{t-1}, \theta_{\tau,t}\right)$, let

$$V_{\tau,t}^{\Omega}\left(\theta_{\tau,t}; \hat{\theta}_{\tau,\tau}^{t-1}\right) = \int_{\underline{\theta}}^{\theta_{\tau,t}} \mathbb{E}\left[\sum_{s=t}^{T} \delta^{s-t} J_t^s\left(\tilde{\theta}_{\tau,t}^s\right) q_{\tau,s}^*\left(\hat{\theta}_{\tau,\tau}^{t-1}, \tilde{\theta}_{\tau,t}^s\right) |\tilde{\theta}_{\tau,t} = r\right] dr.$$

We then specify the transfers $p_{\tau,t}^*$ which deliver these payoffs; i.e., we take

$$p_{\tau,t}^*\left(\hat{\theta}_{\tau,\tau}^{t-1}, \theta_{\tau,t}\right) = \theta_{\tau,t} q_{\tau,t}^*\left(\hat{\theta}_{\tau,\tau}^{t-1}, \theta_{\tau,t}\right) - V_{\tau,t}^{\Omega}\left(\theta_{\tau,t}; \hat{\theta}_{\tau,\tau}^{t-1}\right) + \delta \mathbb{E}\left[V_{\tau,t+1}^{\Omega}\left(\tilde{\theta}_{\tau,t+1}; \hat{\theta}_{\tau,\tau}^{t-1}, \theta_{\tau,t}\right) |\theta_{\tau,t}\right].$$

We now verify that the proposed mechanism is incentive compatible. First note that, since the allocations $q_{\tau,t}^*(\cdot)$ are non-decreasing, Condition (iv) of Corollary 1 in Pavan, Segal and Toikka (2014) is satisfied, so the buyer must be willing to report his values truthfully conditional on participation. This implies that the buyer's expected payoff when participating at any date τ with a value $\theta_{\tau,\tau}$ is equal to

$$V_{\tau,\tau}^{\Omega}(\theta_{\tau,\tau}; \emptyset) = V_{\tau,\tau}^{\Omega}(\underline{\theta}; \emptyset) + \int_{\underline{\theta}}^{\theta_{\tau,\tau}} \mathbb{E}\left[\sum_{s=\tau}^{T} \delta^{s-\tau} J_\tau^s\left(\tilde{\theta}_{\tau,\tau}^s\right) q_{\tau,s}^*\left(\tilde{\theta}_{\tau,\tau}^s\right) |\tilde{\theta}_{\tau,\tau} = r\right] dr.$$

We use this in the remaining step, which is to check the incentive compatibility of immediate participation at the buyer's arrival date.

By the one-shot deviation principle, it suffices to verify that the buyer is willing to participate at an arbitrary date τ. By delaying participation until the following period, the buyer expects a payoff, given $\theta_{\tau,\tau}$, of

$$\mathbb{E}\left[V_{\tau+1,\tau+1}^{\Omega}\left(\tilde{\theta}_{\tau,\tau+1}; \emptyset\right) \mid \tilde{\theta}_{\tau,\tau} = \underline{\theta}\right] + \int_{\underline{\theta}}^{\theta_{\tau,\tau}} \mathbb{E}\left[\sum_{s=\tau+1}^{T} \delta^{s-\tau} J_\tau^s\left(\tilde{\theta}_{\tau,\tau}^s\right) q_{\tau+1,s}^*\left(\tilde{\theta}_{\tau,\tau}^s\right) |\tilde{\theta}_{\tau,\tau} = r\right] dr.$$

By construction,

$$V_{\tau,\tau}^{\Omega}(\underline{\theta}; \emptyset) = \delta \mathbb{E}\left[V_{\tau+1,\tau+1}^{\Omega}\left(\tilde{\theta}_{\tau,\tau+1}; \emptyset\right) \mid \tilde{\theta}_{\tau,\tau} = \underline{\theta}\right].$$

Therefore, a one-shot deviation at date τ to delaying participation is unprofitable if

$$\begin{aligned} & \int_{\underline{\theta}}^{\theta_{\tau,\tau}} \mathbb{E}\left[\sum_{s=\tau}^{T} \delta^{s-\tau} J_\tau^s\left(\tilde{\theta}_{\tau,\tau}^s\right) q_{\tau,s}^*\left(\tilde{\theta}_{\tau,\tau}^s\right) |\tilde{\theta}_{\tau,\tau} = r\right] dr \\ \geq \; & \int_{\underline{\theta}}^{\theta_{\tau,\tau}} \mathbb{E}\left[\sum_{s=\tau+1}^{T} \delta^{s-\tau} J_\tau^s\left(\tilde{\theta}_{\tau,\tau}^s\right) q_{\tau+1,s}^*\left(\tilde{\theta}_{\tau,\tau}^s\right) |\tilde{\theta}_{\tau,\tau} = r\right] dr. \end{aligned}$$

To see that this holds, we reason as follows. First consider our assumption that, for all τ and all $\theta_{\tau,\tau}^{\tau+1}$, $m_{\tau+1}^{\tau+1}(\theta_{\tau,\tau+1}) \leq m_\tau^{\tau+1}\left(\theta_{\tau,\tau}^{\tau+1}\right)$. This implies that, for all τ, all $s > \tau$, and all $\theta_{\tau,\tau}^s$, $m_{\tau+1}^s\left(\theta_{\tau,\tau+1}^s\right) \leq m_\tau^s\left(\theta_{\tau,\tau}^s\right)$, and hence $q_{\tau+1,s}^*\left(\theta_{\tau,\tau+1}^s\right) \leq q_{\tau,s}^*\left(\theta_{\tau,\tau}^s\right)$. That the inequality holds is then immediate from the assumption that J_τ^s is non-negative (equivalently, that the distribution of values after date τ are ordered in the sense of first-order stochastic dominance). ■

Proof of Example 1 . For any τ, t and $(\theta_{\tau,t-1}, \varepsilon_{t-\tau})$, let $z(\theta_{\tau,t-1}, \varepsilon_{t-\tau}) = \bar{\theta}\left(1 - \varepsilon_{t-\tau} e^{-\phi\theta_{\tau,t-1}}\right)$. For any sequence of values $\theta_{\tau,\tau}^t$, we may find for each $s \in \{\tau+1, \ldots, t\}$ the shock $\varepsilon_{s-\tau}$ such that $\theta_{\tau,s} = \bar{\theta}\left(1 - \varepsilon_{s-\tau} e^{-\phi\theta_{\tau,s-1}}\right)$. Indeed, these are given by

$$\varepsilon_{s-\tau} = \frac{\bar{\theta} - \theta_{\tau,s}}{\bar{\theta} e^{-\phi\theta_{\tau,s-1}}}.$$

The chain rule yields that

$$J_\tau^t\left(\theta_{\tau,\tau}^t\right) = \Pi_{s=\tau+1}^t \frac{\partial z\left(\theta_{\tau,s-1}, \varepsilon_{s-\tau}\right)}{\partial \theta_{\tau,s-1}}$$

Hence,

$$\begin{aligned} J_\tau^t\left(\theta_{\tau,\tau}^t\right) &= \Pi_{s=\tau+1}^t \bar{\theta}\phi\varepsilon_{s-\tau} e^{-\phi\theta_{\tau,s-1}} \\ &= \phi^{t-\tau}\Pi_{s=\tau+1}^t\left(\bar{\theta} - \theta_{\tau,s}\right). \end{aligned}$$

Note also that, for $\theta \in \left[0, \bar{\theta}\right]$, $F_{Tr}(\theta|0) = \frac{\bar{\theta}-\theta}{\bar{\theta}}$. Substituting in (13) yields (14). It is then easy to see that Condition (ii) of Proposition 3 is satisfied. That Condition (i) is satisfied follows from (15). ■

Proof of Example 2. Condition (ii) is simple to check. For Condition (i), note that, for each possible θ_1^2,

$$\begin{aligned} \frac{1 - F_{In}(\theta_{1,2})}{f_{In}(\theta_{1,2})} &= \bar{\theta} - \theta_{1,2} \\ &\geq \bar{\theta} - \left(\gamma\theta_{1,1} + \bar{\theta}(1-\gamma)\right) \\ &= \gamma\left(\bar{\theta} - \theta_{1,1}\right) \\ &= \gamma\frac{1 - F_{In}(\theta_{1,1})}{f_{In}(\theta_{1,1})}. \end{aligned}$$

Therefore, $m_2^2(\theta_{1,2}) \leq \theta_{1,2} - \frac{1-F_{In}(\theta_{1,2})}{f_{In}(\theta_{1,2})} \leq \theta_{1,2} - \gamma\frac{1-F_{In}(\theta_{1,1})}{f_{In}(\theta_{1,1})} = m_1^2\left(\theta_{1,1}^2\right)$, as required. ■

Proof of Proposition 4. We begin with Part (i). Suppose for a contradiction that there exists $\varepsilon > 0$ such that, for all $\bar{\tau}$, there is some $\tau > \bar{\tau}$ with $V_{\tau,\tau}(\underline{\theta}; \emptyset) > \varepsilon$. We consider excluding the buyer after date τ, and then reducing the buyer's rents in case of arrival at date τ or before. We argue that this is possible in such a way that the reduction in (ex-ante) expected buyer rents exceeds the (ex-ante) loss in surplus.

Let $\bar{S} = \max_q\{\bar{\theta}q - c(q)\}$ be the upper bound on the surplus that is generated in each period. The total (discounted life-time) surplus generated by a buyer who participates at some date s is no greater than $\overline{SL} = \frac{\bar{S}}{1-\delta}$ in date-s dollars. The contribution to ex-ante expected discounted surplus from arrival after date τ is therefore no greater than

$$\delta^{\tau-1}\sum_{s=\tau+1}^{\infty}\rho_s\delta^{s-\tau}\overline{SL} \leq \delta^{\tau}\overline{SL}\sum_{s=\tau+1}^{\infty}\rho_s$$
$$= \frac{\bar{S}\delta^{\tau}}{1-\delta}\sum_{s=\tau+1}^{\infty}\rho_s.$$

Let $R_\tau = V_{\tau,\tau}(\underline{\theta};\emptyset)$ denote the rent expected conditional on arrival at date τ with value $\underline{\theta}$. Consider excluding participation in the mechanism after date τ and charging an additional participation fee equal to $\delta^{\tau-t}R_\tau$ in case of arrival at each date $t \leq \tau$. The adjusted mechanism remains incentive compatible and induces immediate participation whenever the buyer arrives at date τ or earlier. The reduction in the ex-ante expected rent left to the buyer is at least $\delta^{\tau-1}R_\tau\sum_{s=1}^{\tau}\rho_s$. The increase in profits is therefore at least

$$\delta^{\tau-1}R_\tau\sum_{s=1}^{\tau}\rho_s - \frac{\bar{S}\delta^{\tau}}{1-\delta}\sum_{s=\tau+1}^{\infty}\rho_s$$
$$= \delta^{\tau-1}\left(R_\tau\sum_{s=1}^{\tau}\rho_s - \frac{\bar{S}\delta}{1-\delta}\sum_{s=\tau+1}^{\infty}\rho_s\right). \tag{18}$$

By assumption, we can pick τ arbitrarily large and such that $R_\tau = V_{\tau,\tau}(\underline{\theta};\emptyset) > \varepsilon$. Hence, the expression (18) can be assured strictly positive for τ chosen sufficiently large. That is, profits are higher under the new mechanism.

Now consider Part (ii). If this result does not hold, then there exists $\varepsilon > 0$ such that we can find a sequence $(\tau_k)_{k=1}^{\infty}$ with the property that $\mathbb{E}\left[V_{\tau_k,\tau_k}\left(\tilde{\theta}_{\tau_k,\tau_k};\emptyset\right)\right] \geq \varepsilon$ for all $k = 1, 2, \ldots$. First note that $V_{\tau_k,\tau_k}\left(\theta_{\tau_k,\tau_k};\emptyset\right)$ is uniformly bounded over k and $\theta_{\tau_k,\tau_k} \in \Theta$. Otherwise, by Lemma 4 and the assumption that $q \leq \bar{q}$, we must have $V_{\tau_k,\tau_k}\left(\underline{\theta};\emptyset\right)$ is not uniformly bounded, contradicting Part (i) of the proposition. Because $\mathbb{E}\left[V_{\tau_k,\tau_k}\left(\tilde{\theta}_{\tau_k,\tau_k};\emptyset\right)\right] \geq \varepsilon$ for all k, and because $V_{\tau_k,\tau_k}\left(\cdot;\emptyset\right)$ is non-decreasing by Lemma 4, we can hence find $\nu, \kappa > 0$ such that $V_{\tau_k,\tau_k}\left(\theta_{\tau_k,\tau_k};\emptyset\right) > \nu$ for all $\theta_{\tau_k,\tau_k} > \bar{\theta} - \kappa$. The assumption that $F_{Tr}\left(\cdot|\underline{\theta}\right)$ has full support on Θ then implies the existence of χ such that $\mathbb{E}\left[V_{\tau_k,\tau_k}\left(\tilde{\theta}_{\tau_k-1,\tau_k};\emptyset\right)|\tilde{\theta}_{\tau_k-1,\tau_k-1} = \underline{\theta}\right] > \chi$ for all k. Since $V_{\tau_k-1,\tau_k-1}\left(\underline{\theta};\emptyset\right) \geq \delta\mathbb{E}\left[V_{\tau_k,\tau_k}\left(\tilde{\theta}_{\tau_k-1,\tau_k};\emptyset\right)|\tilde{\theta}_{\tau_k-1,\tau_k-1} = \underline{\theta}\right]$ by the incentive constraint (10), we have established that $V_{\tau_k-1,\tau_k-1}\left(\underline{\theta};\emptyset\right)$ remains bounded above $\delta\chi > 0$, again contradicting Part (i) of the proposition. ■