Robust Predictions in Dynamic Screening

Daniel Garrett, Alessandro Pavan and Juuso Toikka

TSE, Northwestern, MIT

March 2017

Dynamic Mechanism Design

- Mechanism Design: procurement, regulation, employment/compensation, taxation, etc...

- Standard model: one-time information, one-time decisions

- Many settings

  - **information arrives over time**: information is **serially correlated**

  - **sequence of decisions**

- Dynamic mechanism design (DMD)

Dynamic Mechanism Design: Example

- Manufacturer (principal) procures input from supplier (agent)
  - Agent has private information on cost (of quality/quantity each period) that evolves stochastically with time

Standard approach: "relaxed approach"

- How to solve for principal's **profit-maximizing** mechanism?
- Usual approach solves a "**relaxed program**"
  - Account only for certain necessary conditions for IC
    - (derived from "**local**" IC constraints)
  - Verify ex-post that solution to relaxed program is in fact **globally** IC
- Relaxed approach (when it works) often permits complete characterization of optimal mechanism
  - In some cases, optimal mechanism derived in closed form
  - Mimics tractability of Myerson's (1981) optimal auctions work

Existing predictions under stringent conditions

- All predictions in literature to date hinge on validity of the "relaxed approach"
- Relaxed approach works under restrictive conditions on evolution of agent information
  - See Battaglini and Lamba (2015)
  - Restrictive conditions derived by "reverse engineering": not reflective of economic considerations
- Central prediction of existing literature: "**vanishing distortions**"
  - Optimal (profit-maximizing) allocations converge to first-best with time
  - E.g.: Besanko (1985), Battaglini (2005), Pavan-Segal-Toikka (2014), Battaglini and Lamba (2015), Bergemann and Strack (2015)

Our focus: Do vanishing distortions hold more generally?

- Consider broad class of processes where agent types are not fully persistent
  - In particular: Regular Markov processes (have a steady state)
- **"Vanishing distortions" a general feature of dynamic contracting?**
- Key idea: Distortions in allocations to reduce information rents earned by agent due to private information at time of contracting
  - If private information is less than fully persistent, then distortions later in the relationship have little effect on information rents

Contribution: Two positive answers (a different approach)

- Approach: Identify **admissible perturbations** to any IC and IR mechanism
  - Such perturbations should not increase profits ($\longrightarrow$ necessary conditions for optimality)
- **Finding 1:** *Expected* marginal benefit approaches *expected* marginal cost as relationship progresses
  - Holds for any discount factor
- **Finding 2:** Allocations approach efficient ones in probability as relationship progresses
  - Holds for sufficiently large discount factors (or processes that are not too persistent)

MODEL

Dynamic Environment

- Players: Procurer (principal) and supplier (agent)
- $t = 1, 2, \ldots$ (some results also available for finite horizon)
- q_t supplied by supplier to procurer at each date t; total payment x_t
- Common discount factor $\delta \in (0, 1)$

Dynamic Environment

- Principal values quality according to $B : (0, \bar{q}) \to \mathbb{R}$.
  - *Strictly increasing* and *strictly concave*, twice-continuously differentiable function satisfying

$$\lim_{q \searrow 0} B(q) = -\infty.$$

- Agent cost of quality $h_t q_t + C(q_t)$
  - $C(\cdot) : (0, \bar{q}) \to \mathbb{R}_+$ a *strictly increasing* and *strictly convex*, twice-continuously differentiable function satisfying

$$\lim_{q \nearrow \bar{q}} C(q) = +\infty$$

  - Agent date-t "type", $h_t \in \Theta = \{\theta_0, \ldots, \theta_N\}$
    - $0 < \theta_0 < \cdots < \theta_N$

Dynamic Environment

- Principal intertemporal payoff

$$U^P = \sum_{t \geq 1} \delta^{t-1} (B(q_t) - x_t)$$

- Supplier intertemporal payoff

$$U^A = \sum_{t \geq 1} \delta^{t-1} (x_t - h_t q_t - C(q_t))$$

  - Agent outside option equal to zero

- h_t privately observed by agent at the beginning of period t

- Fairly canonical procurement set-up (inspired, e.g., by Baron-Myerson, 1982)

Types process

- Evolution of types governed by *regular* Markov process
  - (Markov chain is aperiodic and irreducible)
- Time-invariant (exogenous) transition matrix A (element ij, prob. of reaching j from i)
  - (Regularity: A^{τ} has only positive elements for some integer $\tau \geq 1$)
- Existence of a unique limiting/stationary distribution $\mathbf{a}^S \in (0,1)^{N+1}$
- Initial types according to a vector of probabilities $\mathbf{a}^1 \in (0,1)^{N+1}$
  - Not necessarily equal to $\mathbf{a}^S$

Mechanisms

- Direct mechanism: a collection of functions $\mathcal{M} = \langle q_t(h^t), p_t(h^t)\rangle_{t=1}^{\infty}$, where $h^t = (h_1, h_2, \ldots, h_t) \in \Theta^t$ represents types up to date t

- For each h^t, let $x_t = p_t(h^t) + C(q_t(h^t))$
    - Agent is always reimbursed $C(q_t(h^t))$ and receives *additional* payment $p_t(h^t)$

Incentive compatibility

- **IC**: Markov process plus discounting $\longrightarrow$ enough to check one-stage deviations from truth-telling

- Agent payoff from date-t onwards under truth-telling

$$V_t\left(h^t\right) = \mathbb{E}\left[\sum_{s=t}^{\infty} \delta^{s-t}\left(p_s\left(\tilde{h}^s\right) - \tilde{h}_s q_s\left(\tilde{h}^s\right)\right)|h^t\right]$$

- **IC**: requirement that, for all t, all $\left(h^{t-1}, h_t\right) \in \Theta^t$, all $h_t' \in \Theta$,

$$V_t\left(h^{t-1}, h_t\right) \geq \mathbb{E}\left[\sum_{s=t}^{\infty} \delta^{s-t}\left(\begin{array}{c} p_s\left(h^{t-1}, h_t', \tilde{h}_{>t}^s\right) \\ -\tilde{h}_s q_s\left(h^{t-1}, h_t', \tilde{h}_{>t}^s\right)\end{array}\right)|h^t\right]$$

Efficient mechanism

- The efficient allocation is $q^E(\cdot)$ s.t. $B'\left(q^E(h_t)\right) = h_t + C'\left(q^E(h_t)\right)$ for all $h_t \in \Theta$
    - One possible efficient (IC) mechanism is obtained by letting $p^E(h_t) + C\left(q^E(h_t)\right) = B\left(q^E(h_t)\right)$ at each t
    - (Here, ignoring participation constraints; i.e., assuming agent willing to participate)

Principal's problem

- Date-1 **participation** constraint

$$V_1(h_1) \geq 0 \text{ for all } h_1 \in \Theta.$$

- Principal seeks to maximize by choice of $\langle q_t(h^t), p_t(h^t)\rangle_{t=1}^{\infty}$ its expected payoff

$$E\left[\sum_{t\geq 1} \delta^{t-1}\left(B\left(q_t\left(\tilde{h}^t\right)\right) - C\left(q_t\left(\tilde{h}^t\right)\right) - p_t\left(\tilde{h}^t\right)\right)\right]$$

subject to incentive and (date-1) participation constraints.

- Let $\mathcal{M}^* = \langle q_t^*(h^t), p_t^*(h^t)\rangle_{t=1}^{\infty}$ denote a solution to principal's problem

ANALYSIS

Convergence in expectation

Proposition

An optimal mechanism $\mathcal{M}^*$ exists, with the allocation rule $q_t^*\left(h^t\right)$ uniquely determined. It satisfies
$\mathbb{E}\left[B'\left(q_t^*\left(\tilde{h}^t\right)\right)\right] \to \mathbb{E}\left[\tilde{h}_t + C'\left(q_t^*\left(\tilde{h}^t\right)\right)\right]$ *as* $t \to +\infty$.

- While efficiency calls for marginal benefit $B'\left(q^E\left(h_t\right)\right)$ equal to marginal cost $h_t + C'\left(q^E\left(h_t\right)\right)$, convergence here is only with respect to *ex-ante expectations*.
- Result leaves the possibility that distortions persist in the long run
    - (Although, convergence to efficient allocations in probability if distortions are always in one direction, say downwards)

Proof (sketch): An admissible perturbation

- Suppose $\langle q_t^* (h^t), p_t^* (h^t)\rangle_{t=1}^{\infty}$ is an optimal (hence IC and IR) policy
  - Note all values $q_t^* \left(h^t\right)$ are interior to $(0, \bar{q})$.
- Considering increase $q_t^* (\cdot)$ uniformly at arbitrary date t, say by small amount $\nu > 0$
  - Recall that the additional cost $C\left(q_t^* \left(h^t\right) + \nu\right) - C\left(q_t^* \left(h^t\right)\right)$ will be reimbursed at date t
- Increase date-1 payments $p_1^* (\cdot)$ uniformly by $\delta^{t-1}\nu \max_{h_1\in\Theta} \mathbb{E}\left[\tilde{h}_t|h_1\right]$
- The new mechanism is incentive compatible and date-1 individually rational
  - It leaves initial type $\hat{h}_1$ an additional expected rent equal to $\delta^{t-1}\nu\left(\max_{h_1\in\Theta} \mathbb{E}\left[\tilde{h}_t|h_1\right] - \mathbb{E}\left[\tilde{h}_t|\hat{h}_1\right]\right)$

Proof (sketch): Proof by contradiction

- If the result is not true, there exists a subsequence of dates (t_n) s.t.

$$\mathbb{E}\left[B'\left(q^*_{t_n}\left(\tilde{h}^{t_n}\right)\right)-\left(\tilde{h}_{t_n}+C'\left(q^*_{t_n}\left(\tilde{h}^{t_n}\right)\right)\right)\right]>\kappa$$

  for all dates t_n in the sequence; or, s.t. $\mathbb{E}\left[B'\left(q^*_{t_n}\left(\tilde{h}^{t_n}\right)\right)-\left(\tilde{h}_{t_n}+C'\left(q^*_{t_n}\left(\tilde{h}^{t_n}\right)\right)\right)\right]<-\kappa$ for an appropriate $\kappa>0$.

- First case (as an example). Increase $q^*_{t_n}(\cdot)$ uniformly at arbitrary date t_n, by arbitrarily small amount $\nu_n>0$
  - (ν_n can be chosen small enough, depending on n, such that the new policy remains interior).

- Adjust payments as described above, leaving new mechanism IC and (date-1) IR

Proof (sketch)

- The new mechanism increases expected surplus by at least $\delta^{t_n - 1}\kappa\nu_n$ (ν_n small enough)

- The new mechanism leaves an additional expected rent to an agent whose date-1 type is $\hat{h}_1$ equal to

$$\delta^{t_n - 1}\nu_n \left(\max_{h_1} \left\{ \mathbb{E}\left[\tilde{h}_{t_n} | h_1 \right] \right\} - \mathbb{E}\left[\tilde{h}_{t_n} | \hat{h}_1 \right] \right).$$

- Since, for all $\hat{h}_1 \in \Theta$,

$$\max_{h_1} \left\{ \mathbb{E}\left[\tilde{h}_{t_n} | h_1 \right] \right\} - \mathbb{E}\left[\tilde{h}_{t_n} | \hat{h}_1 \right] \to 0 \text{ as } t_n \to \infty$$

the increase in surplus dominates for t_n large enough.

Convergence to efficiency in probability

Proposition

Fix the stochastic process and technology. There exists $\bar{\delta}$ such that, for any $\delta \in (\bar{\delta}, 1)$, the following are true:

1. $\lim_{t\to\infty} \mathbb{E}\left[B\left(q_t^*\left(\tilde{h}^t\right)\right) - \left(\tilde{h}_t q_t^*\left(\tilde{h}^t\right) + C\left(q_t^*\left(\tilde{h}^t\right)\right)\right)\right]$
$= \lim_{t\to\infty} \mathbb{E}\left[B\left(q^E\left(\tilde{h}_t\right)\right) - \left(\tilde{h}_t q^E\left(\tilde{h}_t\right) + C\left(q^E\left(\tilde{h}_t\right)\right)\right)\right].$
2. *For any* $\eta > 0$, $\lim_{t\to\infty} \Pr\left(\left|q_t^*\left(\tilde{h}^t\right) - q^E\left(\tilde{h}_t\right)\right| > \eta\right) = 0.$

1. Expected loss in surplus from distortions vanishes in the long run
2. Optimal allocation converges to efficient one in probability

Proof (sketch): Efficient mechanism

- Approach focuses on combining putative optimal mechanism $\mathcal{M}^*$ with efficient dynamic mechanism
- Efficient dynamic mechanism: To fix ideas $p^E(h_t) = B\left(q^E(h_t)\right) - C\left(q^E(h_t)\right)$ at each t, $h_t \in \Theta$
  - This is a *repetition of static mechanisms*
  - Continuation of efficient mechanism thus unaffected by current reports
    - IC constraints equivalent to static ones

Proof (sketch): Going to efficiency too fast

- **Idea 1 (FAILED!):** Suppose $\mathcal{M}^* = \langle q_t^* \left(h^t\right), p_t^* \left(h^t\right)\rangle_{t=1}^{\infty}$ is an optimal mechanism such that result does not hold.
  - Replace the payment and allocation rules with those of the efficient mechanism from date t onwards.
  - Adjust payments, to ensure satisfaction of date-1 participation constraints.
  - Such an adjustment can be made s.t. (expected) increase in surplus dominates any increases in information rents (when t large enough)
    - *Assuming new mechanism is IC*
- **Problem:** New mechanism need not be IC.
  - IC from date t onwards, but not necessarily at earlier dates.

Proof (sketch): Approaching efficiency gradually

- **Idea 2 (WORKS!):** *Approach efficiency gradually*

- Observation A: Efficient mechanism has a fixed amount of slack at all histories
  - Exists $\omega > 0$ s.t., for any θ_j, θ_k with $\theta_j \neq \theta_k$,

$$\begin{aligned} \begin{pmatrix} p^E(\theta_j) - \theta_j q^E(\theta_j) \\ -\left(p^E(\theta_k) - \theta_j q^E(\theta_k)\right) \end{pmatrix} &\geq \omega\,|\theta_j - \theta_k| \\ &\geq \omega \min\{\theta_j - \theta_k : \theta_j \neq \theta_k\}. \end{aligned}$$

- Observation B: For any $\tau \geq 1$, any linear convex combination of $\langle q_t^*(h^t), p_t^*(h^t)\rangle_{t=\tau}^{\infty}$ and $\langle q^E(h_t), p^E(h_t)\rangle_{t=\tau}^{\infty}$ satisfies IC at all histories h^{t-1} for $t \geq \tau$.
  - At least some fixed amount of slack in all IC constraints (according to ω and linear weights)

Proof (sketch): Approaching efficiency gradually

- Observations permit gradual growth in weight on efficiency
  - For example, put

$$\begin{aligned}\left(q_1^{new}(h_1), p_1^{new}(h_1)\right) &= \left(1-\alpha^1\right)\left(q_1^*(h_1), p_1^*(h_1)\right) \\ &\quad +\alpha^1\left(q^E(h_1), p^E(h_1)\right), \text{ and} \\ \left(q_t^{new}(h^t), p_t^{new}(h^t)\right) &= \left(1-\alpha^{\geq 2}\right)\left(q_t^*(h^t), p_t^*(h^t)\right) \\ &\quad +\alpha^{\geq 2}\left(q^E(h_t), p^E(h_t)\right)\end{aligned}$$

  for $t \geq 2$, where $0 < \alpha^1 \leq \alpha^{\geq 2} \leq 1$.

- The new mechanism is IC if $\alpha^{\geq 2}$ is not too much larger than α^1.
  - For fixed $\alpha^{\geq 2}$, mechanism is IC from date 2 onwards.
  - If $\alpha^1 = \alpha^{\geq 2}$, then there is slack in IC at date 1. Hence, can decrease α^1 below $\alpha^{\geq 2}$ by a small amount.

Proof (sketch): Condition on discount factor

- **Why we need δ close to 1?**
  - Proposed "new" mechanism approaches efficiency gradually.
  - Positive weight on efficient policy at early dates may increase information rents (by relatively large amount)
  - When δ is small, gains in surplus at later dates need not exceed these increased rents.

Corollary: any discount factor and small persistence

Proposition

Fix the technology, initial type distribution $\mathbf{a}^1$*, and discount factor* δ*. There exists* $\bar{m}$ *s.t., for all* $m \geq \bar{m}$*, if the transition matrix is* A^m*:*

1. $$\lim_{t\to\infty} \mathbb{E}\left[B\left(q_t^*\left(\tilde{h}^t\right)\right) - C\left(q_t^*\left(\tilde{h}^t\right)\right) - \tilde{h}_t q_t^*\left(\tilde{h}^t\right)\right]$$
   $$= \lim_{t\to\infty} \mathbb{E}\left[B\left(q^E\left(\tilde{h}_t\right)\right) - C\left(q_t^E\left(\tilde{h}_t\right)\right) - \tilde{h}_t q^E\left(\tilde{h}_t\right)\right].$$
2. *For any* $\eta > 0$, $\lim_{t\to\infty} \Pr\left(\left|q_t^*\left(\tilde{h}^t\right) - q^E\left(\tilde{h}_t\right)\right| > \eta\right) = 0.$

- I.e., convergence to efficiency holds for any $\delta \in (0,1)$ provided process is not very persistent
- Related: Convergence always holds if principal and agent meet (sufficiently) infrequently (say every m periods)

CONTINUOUS TYPES

Continuous types: Stochastic process

- Suppose date-t type h_t is now drawn according to time-invariant Markov chain $F = (F_t)$; $\Theta = [\underline{\theta}, \bar{\theta}] \subset \mathbb{R}_+$
- F_1 is (abs. continuous) cdf of initial distribution; density f_1
- $F_t(\cdot|h_{t-1})$ cdf of h_t, given $h_{t-1} \in \Theta$ ($F_t(\cdot|h_{t-1})$ full support on Θ)
- *Stochastic monotonicity*: $F_t(\cdot|h'_{t-1})$ first-order stochastically dominates $F_t(\cdot|h_{t-1})$ for $h'_{t-1} > h_{t-1}$
- *Time-invariance*: $F_t(\cdot|\theta) = F_s(\cdot|\theta)$ all $t, s > 1$, all $\theta \in \Theta$
- *Stationarity assumption*: $\exists!$ invariant distribution π s.t., for all $\theta \in \Theta$

$$\sup_{A \in \mathcal{B}(\Theta)} \left| F^t(A; \theta) - \pi(A) \right| \to 0 \text{ as } t \to \infty.$$

Continuous types: Auxiliary shock representation

- Stochastic process can be represented via "auxiliary shocks"
    - Shocks independent of initial private information
    - Follow Eso, Szentes (2007), Pavan, Segal, Toikka (2014)
- $h_t = z(h_{t-1}, \varepsilon_t)$, where $\varepsilon = (\varepsilon_t)$ are i.i.d. random variables
- E.g., take ε_t to have uniform distribution, and determine z via "probability integral transform" result
- Assume additionally that $\frac{\partial z(h_{t-1}, \varepsilon_t)}{\partial h_{t-1}}$ exists and is continuous and bounded

Continuous types: Impulse responses

- Let $(Z_{\tau,t})_{t\geq\tau}$ a collection of functions s.t. $h_t = Z_{\tau,t}(h_\tau, \varepsilon)$ for $t \geq \tau$
- **Impulse responses**:

$$I_{\tau\to t}\left(h^t\right) = \frac{\partial Z_{\tau,t}(h_\tau, \varepsilon)}{\partial h_\tau}$$

(where vector ε derived from h^t using function z)

- AR(1) example:

$$\begin{aligned} h_t &= \gamma h_{t-1} + \varepsilon_t \\ &= Z_{\tau,t}(h_\tau, \varepsilon) = \gamma^{t-\tau} h_\tau + \gamma^{t-\tau-1}\varepsilon_{\tau+1} + \cdots + \gamma\varepsilon_{t-1} + \varepsilon_t \\ \rightarrow\ & I_{\tau\to t}\left(h^t\right) = \gamma^{t-\tau}. \end{aligned}$$

Characterization of IC

Theorem (Pavan, Segal, Toikka, 2014)

Mechanism $\mathcal{M} = \langle q_t(h^t), p_t(h^t)\rangle_{t=1}^{\infty}$ *IC iff, for all* $t \geq 0$*, all* h^{t-1}*,* $V_t(h^t)$ *is Lipschitz continuous in* h_t *with*

$$\frac{\partial V_t(h^t)}{\partial h_t} = -\mathbb{E}\left[\sum_{s\geq t}\delta^{s-t} I_{t\to s}\left(\tilde{h}^s\right) q_s\left(\tilde{h}^s\right)|h^t\right] \text{ a.e.,}$$

and, for all h^{t-1}*,* h_t*,* $\hat{h}_t$*,*

$$\int_{\hat{h}_t}^{h_t}\left[D_t\left(\left(h^{t-1},x\right);x\right) - D_t\left(\left(h^{t-1},x\right);\hat{h}_t\right)\right]dx \geq 0$$

where

$$D_t\left(h^t;y\right) \equiv -\mathbb{E}\left[\sum_{s\geq t}\delta^{s-t} I_{t\to s}\left(\tilde{h}^s\right) q_s\left(\tilde{h}^s_{-t},y\right)|h^t\right].$$

Dynamic virtual surplus

- Use result (and integration by parts) to write "dynamic virtual surplus"

$$\mathbb{E}\left[\sum_{t\geq 1}\delta^{t-1}\left(\begin{array}{c} B\left(q_t\left(\tilde{h}^t\right)\right)-C\left(q_t\left(\tilde{h}^t\right)\right) \\ -q_t\left(\tilde{h}^t\right)\left(\tilde{h}_t+\frac{F_1\left(\tilde{h}_1\right)}{f_1\left(\tilde{h}_1\right)}I_{1\to t}\left(\tilde{h}^t\right)\right) \end{array}\right)\right]$$
$$-V_1\left(\bar{\theta}\right)$$

(FOSD ensures IC binds at $\bar{\theta}$, so can put $V_1\left(\bar{\theta}\right)=0$ for an optimum).

Relaxed approach

- "**Relaxed approach**"/"**First-order approach**": Pointwise maximization

$$B'\left(q_t^*\left(h^t\right)\right) = C'\left(q_t^*\left(h^t\right)\right) + h_t + \frac{F_1\left(h_1\right)}{f_1\left(h_1\right)} I_{1\to t}\left(h^t\right)$$

- FOSD ($I_{1\to t} \geq 0$) implies distortions always downwards
    - Distortions driven by impulse responses
    - Solution must be checked against "integral monotonicity" constraint

Convergence in expectation

Proposition

An optimal policy exists, with the optimal allocation rule $\left(q_t^*\left(h^t\right)\right)_{t=1}^{\infty}$ *uniquely determined. Suppose that, for each* t, $0 < \underline{\iota}_t \leq q_t^*\left(h^t\right) \leq \overline{\iota}_t < \bar{q}$ *for history-invariant constants* $\underline{\iota}_t$ *and* $\overline{\iota}_t$. *Then the optimal policy satisfies*

$$\begin{aligned} \mathbb{E}\left[B'\left(q_t^*\left(\tilde{h}^t\right)\right) - C'\left(q_t^*\left(\tilde{h}^t\right)\right) - \tilde{h}_t\right] &= \mathbb{E}\left[\frac{F_1\left(\tilde{h}_1\right)}{f_1\left(\tilde{h}_1\right)} I_{1\to t}\left(\tilde{h}^t\right)\right] \\ &\to 0 \text{ as } t \to +\infty. \end{aligned}$$

Provided $F_1 = \pi$, *convergence is monotone.*

Convergence in expectation: Comments

- With continuum of types, we are (to date) unable to guarantee that optimal policy $q_t^* \left(h^t\right)$ remains bounded away from the boundary (0 and $\bar{q}$)

- However, if this holds, we can quantify expected distortions (gap between MB and MC) for any t
  - Interiority could be guaranteed (result applies) if restrict $q_t\left(\cdot\right)$ to be Lipschitz (with fixed but large Lipschitz constant)

In progress...

- Version of convergence to efficiency for
  - Allocations restricted to be Lipschitz (with fixed, but large constant)
  - δ sufficiently large (for fixed stochastic process and Lipschitz constant)

CONCLUSIONS

Conclusions

- Earlier literature derived convergence to efficient allocations applying relaxed/first-order approach
  - Heavily restricted classes of processes
- This paper: Sufficient conditions for convergence to efficiency
  - Discrete types, regular Markov process, sufficient degree of patience
  - Also weaker notion of convergence (distortions vanish in expectation)
  - Built on Garrett/Pavan (2015, JET): 2 periods with risk averse agent
- Results seem to confirm observations made with relaxed/first-order approach